

MAGNETIC SCHRÖDINGER OPERATORS AND MAÑÉ'S CRITICAL VALUE

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ABSTRACT. We study periodic magnetic Schrödinger operators on covers of closed manifolds in relation to Mañé's critical energy values of the corresponding classical Hamiltonian systems. In particular, we show that if the covering transformation group is amenable, then the bottom of the spectrum is bounded from above by Mañé's critical energy value. We also determine the spectra for various homogeneous spaces with left-invariant magnetic fields.

1. INTRODUCTION

Ever since quantum mechanics was formulated by Heisenberg and Schrödinger in the 1920s, it has been one of the main objectives of mathematical physics to understand its relations with classical mechanics. We exploit one instance of this interplay and relate Mañé's critical energy value of classical electromagnetic Hamiltonians with the ground state energy of the associated magnetic Schrödinger operators.

More precisely, let M be a connected closed Riemannian manifold that is equipped with an electric potential and a magnetic field given by a smooth function $\mathbf{V} \in C^\infty(M, \mathbb{R})$ and a closed 2-form $\beta \in \Omega^2(M, \mathbb{R})$, respectively. Any regular cover $\widehat{\pi}: \widehat{M} \rightarrow M$, for which $\widehat{\beta} = \widehat{\pi}^*\beta$ has a magnetic potential $\widehat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$ satisfying $\widehat{\beta} = d\widehat{\alpha}$, gives rise to a Hamiltonian $H_{\widehat{\alpha}, \widehat{\mathbf{V}}}: T^*\widehat{M} \rightarrow \mathbb{R}$ defined as $H_{\widehat{\alpha}, \widehat{\mathbf{V}}}(x, p) = \frac{1}{2}|p + \widehat{\alpha}|_x^2 + \widehat{\mathbf{V}}(x)$, where $\widehat{\mathbf{V}} = \widehat{\pi}^*\mathbf{V} = \mathbf{V} \circ \widehat{\pi}$. Mañé's critical value of the corresponding Lagrangian $L_{\widehat{\alpha}, \widehat{\mathbf{V}}}: T\widehat{M} \rightarrow \mathbb{R}$ is given by [Mañ97, CIPP98, BP02]

$$c(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}) = \inf_{f \in C^\infty(\widehat{M}, \mathbb{R})} \sup_{x \in \widehat{M}} \left(\frac{1}{2} |\widehat{\alpha} + df|_x^2 + \widehat{\mathbf{V}}(x) \right).$$

For the sake of convenience, all covers are implicitly assumed to be connected. In Section 2, we generalize results from [PP97, FM07] concerning the exact case in which $\widehat{\alpha} = \widehat{\pi}^*\alpha$ for some $\alpha \in \Omega^1(M, \mathbb{R})$. We prove that any regular cover \widehat{M} of M whose covering transformation group G is amenable satisfies

$$(1.1) \quad c(L_{\widetilde{\alpha}, \widetilde{\mathbf{V}}}) = c(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}) = \min_{[\omega] \in H^1(M, \mathbb{R}) : \widehat{\omega} \text{ is exact}} c(L_{\alpha - \omega, \mathbf{V}}),$$

where $L_{\widetilde{\alpha}, \widetilde{\mathbf{V}}}$, $L_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ and $\widehat{\omega}$ denote the lifts of $L_{\alpha, \mathbf{V}}$ and ω to \widehat{M} and to a subcover \widehat{M} of \widehat{M} whose covering transformation group is isomorphic to the abelianization $G/[G, G]$.

In Section 3, we study the quantum analogue of magnetic Hamiltonians of the form $H_{\widehat{\alpha}, \widehat{\mathbf{V}}}$, that is, magnetic Schrödinger operators $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ initially defined on $C_0^\infty(\widehat{M}, \mathbb{C})$ as

$$\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}u = \frac{1}{2}\Delta u - i\langle du, \widehat{\alpha} \rangle + \left(\frac{1}{2}d^*\widehat{\alpha} + \frac{1}{2}|\widehat{\alpha}|^2 + \widehat{\mathbf{V}} \right)u.$$

It is known [Shu01, BMS02], that for arbitrary $\widehat{\alpha}$ and periodic $\widehat{\mathbf{V}} = \widehat{\pi}^* \mathbf{V}$ as above, the closure $\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}}$ is a self-adjoint operator in $L^2(\widehat{M}, \mathbb{C})$. The main object of study is the ground state energy defined as

$$\lambda_0(\widehat{\alpha}, \widehat{\mathbf{V}}) = \inf \operatorname{spec}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}}).$$

We generalize various results which are either known for $\widehat{\alpha} = 0$ or $\widehat{M} = \mathbb{R}^n$. The exact case allows for a detailed spectral analysis in terms of twisted operators and representations of the covering transformation group of \widehat{M} [Sun89, KOS89]. We moreover prove theorems about amenable and abelian covers, which are motivated by corresponding results for the discrete analogue of $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ on periodic graphs [HS99, HS01]. In particular, abelian covers $\widehat{\pi}: \widehat{M} \rightarrow M$ satisfy

$$(1.2) \quad \lambda_0(\widehat{\alpha}, \widehat{\mathbf{V}}) = \min_{[\omega] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})} \lambda_0(\alpha - \omega, \mathbf{V}),$$

where

$$H^1(M, \widehat{M}, \mathbb{Z}) = \left\{ [\omega] \in H^1(M, \mathbb{R}) \mid \int_{\gamma} \widehat{\omega} \in \mathbb{Z} \text{ for any closed curve } \gamma \text{ in } \widehat{M} \right\}.$$

The structural resemblance of (1.1) and (1.2) motivated the main result in Section 4, which generalizes [Pat01, Theorem B] to non-compact amenable covers as follows.

Theorem 23. *Let \widehat{M} be a regular cover of M with amenable covering transformation group. If $\widehat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$ and $\widehat{\mathbf{V}} \in C^\infty(\widehat{M}, \mathbb{R})$ are potentials with $\inf \widehat{\mathbf{V}} > -\infty$, then the associated Lagrangian $L_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ and the associated magnetic Schrödinger operator $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ satisfy*

$$\lambda_0(\widehat{\alpha}, \widehat{\mathbf{V}}) \leq c(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}).$$

In the last section, we explicitly determine $\operatorname{spec}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}})$ on various covers \widehat{M} of compact homogeneous spaces $M = \Lambda \backslash \Gamma$, which facilitates comparisons with the corresponding classical data. In each case, Γ is a Lie group that is equipped with a left-invariant metric and a left-invariant magnetic field $\beta \in \Omega^2(\Gamma, \mathbb{R})$, and $\Lambda \subset \Gamma$ is a cocompact lattice. The following exact examples have not been studied before and exhibit unexpected phenomena:

- $PSL(2, \mathbb{R})$ has a left-invariant magnetic potential $\widehat{\alpha}$, such that $\lambda_0(\mathbf{B}\widehat{\alpha}, 0) > c(L_{\mathbf{B}\widehat{\alpha}, 0})$ near $\mathbf{B} = 0$, and the mapping $\mathbf{B} \mapsto \lambda_0(\mathbf{B}\widehat{\alpha}, 0)$ has 2 non-trivial local minima. The corresponding classical dynamics are well-understood [CFP10].
- The compact quotient $\Lambda \backslash \operatorname{Nil}$ of the Heisenberg group Nil by the lattice Λ of integer matrices has a magnetic potential α , such that the mapping $\mathbf{B} \mapsto \lambda_0(\mathbf{B}\widehat{\alpha}^{[\Lambda, \Lambda] \backslash \operatorname{Nil}}, 0)$ has countably many local minima, where $\widehat{\alpha}^{[\Lambda, \Lambda] \backslash \operatorname{Nil}}$ denotes the lift of α to the maximal abelian cover $[\Lambda, \Lambda] \backslash \operatorname{Nil}$. This is the first example in which the ground state energy is an unbounded function of the strength of the magnetic field with infinitely many local minima.

Irrespective of amenability of $\pi_1(M)$, all exact examples in Section 5 satisfy

$$\lambda_0(\mathbf{B}\widehat{\alpha}, 0) = \lambda_0(0, 0) + c(L_{\mathbf{B}\widehat{\alpha}, 0}) \quad \text{near } \mathbf{B} = 0$$

on the respective universal covers and various intermediate covers. The planar restricted 3-body problem is touched in the final subsection. It lies beyond the scope of the theory that will be developed in the next sections, and hints at possible further development.

2. MAÑÉ'S CRITICAL VALUE OF MAGNETIC HAMILTONIANS

Let M be a connected closed manifold with smooth Riemannian metric g and associated norms on TM and T^*M denoted by $|v|_x$ and $|p|_x$ for $v \in T_x M$ and $p \in T_x^* M$, respectively. Any pair consisting of a smooth function $\mathbf{V} \in C^\infty(M, \mathbb{R})$ and a closed 2-form $\beta \in \Omega^2(M, \mathbb{R})$ can be viewed as an electromagnetic field acting on charged particles whose motion is confined to M . More precisely, let $\pi: T^*M \rightarrow M$ be the canonical projection and let $\omega_0 = -d\lambda$ be the canonical symplectic form of T^*M with Liouville 1-form λ . The triple (g, β, \mathbf{V}) gives rise to the Hamiltonian system $(T^*M, \omega_\beta, H_\mathbf{V})$ with Hamiltonian $H_\mathbf{V}(x, p) = \frac{1}{2}|p|_x^2 + \mathbf{V}(x)$ and twisted symplectic structure $\omega_\beta = \omega_0 + \pi^*\beta$. The metric g induces the canonical bundle isomorphism $\flat: TM \rightarrow T^*M$, which in turn gives rise to the dual system $(TM, \flat^*\omega_\beta, \flat^*H_\mathbf{V})$. The corresponding Hamiltonian flow on TM is called the electromagnetic flow of (g, β, \mathbf{V}) since its orbits coincide with the trajectories of a particle of unit mass and charge under the influence of the conservative force $\nabla \mathbf{V}$ and the Lorentz force $F: TM \rightarrow TM$ defined by [BP02]

$$\beta_x(v, w) = g_x(F_x(v), w) \quad \text{for all } x \in M \text{ and } v, w \in T_x M.$$

The flow of $(g, \beta, 0)$ is called magnetic flow or twisted geodesic flow [BP02, Pat09] since the triple $(g, 0, 0)$ gives rise to the geodesic flow of g .

In the following, let $\widehat{\pi}: \widehat{M} \rightarrow M$ be a regular cover such that $\widehat{\beta} = \widehat{\pi}^*\beta$ is exact with magnetic potential $\widehat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$, that is, $d\widehat{\alpha} = \widehat{\beta}$. Whenever there exists a magnetic potential, one can remove the twist in the lift $\widehat{\omega}_\beta$ of the symplectic structure ω_β as follows. Let $\widehat{\mathbf{V}}, \widehat{g}, \widehat{\omega}_0$ and $H_{\widehat{\mathbf{V}}}$ denote the lifts of \mathbf{V}, g, ω_0 and $H_\mathbf{V}$, respectively. The flow of the system $(T^*\widehat{M}, \widehat{\omega}_\beta, H_{\widehat{\mathbf{V}}})$ is equivalent to the flow of the system $(T^*\widehat{M}, \widehat{\omega}_0, H_{\widehat{\alpha}, \widehat{\mathbf{V}}})$ with Hamiltonian

$$H_{\widehat{\alpha}, \widehat{\mathbf{V}}}(x, p) = H_{\widehat{\mathbf{V}}}(x, p + \widehat{\alpha}) = \frac{1}{2}|p + \widehat{\alpha}|_x^2 + \widehat{\mathbf{V}}(x).$$

An equivalence is given by the mapping $(x, p) \mapsto (x, p + \widehat{\alpha}_x)$. Mañé's critical value of the Hamiltonian $H_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ is defined as [CFP10]

$$(2.1) \quad \mathbf{c}(H_{\widehat{\alpha}, \widehat{\mathbf{V}}}) = \inf_{\widehat{\alpha}' \in \Omega^1(\widehat{M}, \mathbb{R}): d\widehat{\alpha}' = \widehat{\beta}} \sup_{x \in \widehat{M}} H_{\widehat{\mathbf{V}}}(x, \widehat{\alpha}'_x) = \inf_{\widehat{\omega} \in \Omega^1(\widehat{M}, \mathbb{R}): d\widehat{\omega} = 0} \sup_{x \in \widehat{M}} H_{\widehat{\alpha}, \widehat{\mathbf{V}}}(x, \widehat{\omega}_x).$$

The existence of magnetic potentials allows for a description of $\mathbf{c}(H_{\widehat{\alpha}, \widehat{\mathbf{V}}})$ in terms of the Legendre transform $L_{\widehat{\alpha}, \widehat{\mathbf{V}}}: T\widehat{M} \rightarrow \mathbb{R}$, that is, in terms of the Lagrangian given by

$$(2.2) \quad L_{\widehat{\alpha}, \widehat{\mathbf{V}}}(x, v) = \frac{1}{2}|v|_x^2 - \widehat{\alpha}_x(v) - \widehat{\mathbf{V}}(x).$$

The solutions of the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L_{\widehat{\alpha}, \widehat{\mathbf{V}}}}{\partial v}(x, v) = \frac{\partial L_{\widehat{\alpha}, \widehat{\mathbf{V}}}}{\partial x}(x, v)$$

give rise to the Euler-Lagrange flow, whose orbits are known to coincide with the orbits of the electromagnetic flow of $(\widehat{g}, \widehat{\beta}, \widehat{\mathbf{V}})$. We let $A_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ denote the action of $L_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ on the space of absolutely continuous curves $\gamma: [a, b] \rightarrow \widehat{M}$ given as

$$A_{\widehat{\alpha}, \widehat{\mathbf{V}}}(\gamma) = \int_a^b L_{\widehat{\alpha}, \widehat{\mathbf{V}}}(\gamma(t), \dot{\gamma}(t)) dt.$$

Mañé [Mañ97] defined the critical value of the Lagrangian $L_{\hat{\alpha}, \hat{\mathbf{v}}}$ as

$$\mathbf{c}(L_{\hat{\alpha}, \hat{\mathbf{v}}}) = \inf \left\{ k \in \mathbb{R} \cup \{\infty\} \mid A_{\hat{\alpha}, \hat{\mathbf{v}}+k}(\gamma) \geq 0 \text{ for any closed curve } \gamma \right\}.$$

Burns and Paternain [BP02] gave the following Hamiltonian description of $\mathbf{c}(L_{\hat{\alpha}, \hat{\mathbf{v}}})$, which is a generalization of [CIPP98, Theorem A]

$$\begin{aligned} (2.3) \quad \mathbf{c}(L_{\hat{\alpha}, \hat{\mathbf{v}}}) &= \inf_{f \in C^\infty(\widehat{M}, \mathbb{R})} \sup_{x \in \widehat{M}} H_{\hat{\alpha}, \hat{\mathbf{v}}}(x, d_x f) \\ &= \inf \left\{ k \in \mathbb{R} \cup \{\infty\} \mid \text{there exists } f \in C^\infty(\widehat{M}, \mathbb{R}) : H_{\hat{\alpha}, \hat{\mathbf{v}}}(df) < k \right\}. \end{aligned}$$

In other words, $\mathbf{c}(L_{\hat{\alpha}, \hat{\mathbf{v}}})$ is the infimum of values $k \in \mathbb{R} \cup \{\infty\}$ for which $H_{\hat{\alpha}, \hat{\mathbf{v}}}^{-1}(-\infty, k)$ contains an exact Lagrangian graph. Note that replacing $\hat{\alpha}$ by $\hat{\alpha} + df$ for some $f \in C^\infty(\widehat{M}, \mathbb{R})$ does not effect $\mathbf{c}(L_{\hat{\alpha}, \hat{\mathbf{v}}})$. Hence, another magnetic potential $\hat{\alpha}'$ may yield a different critical value only if $\hat{\alpha} - \hat{\alpha}'$ corresponds to a non-zero cohomology class in $H^1(\widehat{M}, \mathbb{R})$. A comparison of (2.1) and (2.3) leads to

$$(2.4) \quad \mathbf{c}(H_{\hat{\alpha}, \hat{\mathbf{v}}}) = \inf_{[\hat{\omega}] \in H^1(\widehat{M}, \mathbb{R})} \mathbf{c}(L_{\hat{\alpha} - \hat{\omega}, \hat{\mathbf{v}}}).$$

If $L_{\tilde{\alpha}, \tilde{\mathbf{v}}}$ is the lift of $L_{\hat{\alpha}, \hat{\mathbf{v}}}$ to a regular cover \widetilde{M} of \widehat{M} , then

$$(2.5) \quad \mathbf{c}(L_{\tilde{\alpha}, \tilde{\mathbf{v}}}) \leq \mathbf{c}(L_{\hat{\alpha}, \hat{\mathbf{v}}}) \quad \text{and} \quad \mathbf{c}(H_{\tilde{\alpha}, \tilde{\mathbf{v}}}) \leq \mathbf{c}(H_{\hat{\alpha}, \hat{\mathbf{v}}}),$$

with equality if \widetilde{M} is a finite cover of \widehat{M} . In the so-called exact case in which $\beta = d\alpha$ for some $\alpha \in \Omega^1(M, \mathbb{R})$, we have a Hamiltonian $H_{\alpha, \mathbf{v}}$ and a Lagrangian $L_{\alpha, \mathbf{v}}$ on M . Mañé [Mañ97] coined the phrase strict critical value for $\mathbf{c}(H_{\alpha, \mathbf{v}})$, and denoted it by $\mathbf{c}_0(L_{\alpha, \mathbf{v}})$. He related $\mathbf{c}_0(L_{\alpha, \mathbf{v}})$ to Mather's action functional $\alpha: H^1(M, \mathbb{R}) \rightarrow \mathbb{R}$ given by [Mat91]

$$\alpha([\omega]) = - \min \left\{ \int L_{\alpha + \omega, \mathbf{v}} d\mu \mid \mu \in \mathcal{M}(L_{\alpha, \mathbf{v}}) \right\},$$

where $\mathcal{M}(L_{\alpha, \mathbf{v}})$ denotes the set of probabilities on the Borel σ -algebra of TM that have compact support and are invariant under the Euler-Lagrange flow. Mañé [Mañ97] proved that

$$\alpha([\omega]) = \mathbf{c}(L_{\alpha + \omega, \mathbf{v}}).$$

Since α is convex and superlinear [Mat91, Theorem 1], one obtains

$$\mathbf{c}_0(L_{\alpha, \mathbf{v}}) = \mathbf{c}(H_{\alpha, \mathbf{v}}) = \min_{[\omega] \in H^1(M, \mathbb{R})} \alpha([\omega]).$$

We let $\widehat{\pi}^{\text{univ}}: \widehat{M}^{\text{univ}} \rightarrow M$ and $\widehat{\pi}^{\text{abel}}: \widehat{M}^{\text{abel}} \rightarrow M$ denote the universal and the maximal abelian cover of M , respectively. Their covering transformation groups are the fundamental group $\pi_1(M)$ and the first homology group $H_1(M, \mathbb{Z})$. More precisely, $\widehat{M}^{\text{abel}}$ is defined as the regular cover of M whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(M) \rightarrow H_1(M, \mathbb{Z})$, that is, the commutator subgroup $[\pi_1(M), \pi_1(M)]$. Therefore, any regular cover with an abelian covering transformation group is covered by $\widehat{M}^{\text{abel}}$.

In the monopole case in which $[\beta] \neq 0 \in H^2(M, \mathbb{R})$, one can define $L_{\hat{\alpha}, \hat{\mathbf{v}}}^{\text{univ}}: T\widehat{M}^{\text{univ}} \rightarrow \mathbb{R}$, respectively $L_{\hat{\alpha}, \hat{\mathbf{v}}}^{\text{abel}}: T\widehat{M}^{\text{abel}} \rightarrow \mathbb{R}$, as in (2.2) only if the lift of β to $\widehat{M}^{\text{univ}}$, respectively to $\widehat{M}^{\text{abel}}$, is exact. If this is the case, we obtain $\mathbf{c}(H_{\hat{\alpha}, \hat{\mathbf{v}}}^{\text{univ}}) = \mathbf{c}(L_{\hat{\alpha}, \hat{\mathbf{v}}}^{\text{univ}})$ directly from (2.4) since

$H^1(\widehat{M}^{\text{univ}}, \mathbb{R})$ is trivial. In the exact case, it is known that $\mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}^{\text{abel}}) = \mathbf{c}_0(L_{\alpha, \mathbf{V}})$ [PP97]. Recently, Fathi and Maderna [FM07] proved that if, in addition, $\pi_1(M)$ is amenable, then

$$(2.6) \quad \mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}^{\text{univ}}) = \mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}^{\text{abel}}) = \mathbf{c}_0(L_{\alpha, \mathbf{V}}),$$

which also implies $\mathbf{c}(H_{\widehat{\alpha}, \widehat{\mathbf{V}}}^{\text{abel}}) = \mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}^{\text{abel}})$. For the reader's convenience, we recall the notion of amenability.

Definition 1. A discrete group G is called amenable, if there exists a continuous functional m on the space $L^\infty(G, \mathbb{R})$ of bounded real-valued functions on G such that

- (1) $m(1_G) = 1$, where $1_G(\gamma) = 1$ for all $\gamma \in G$,
- (2) if $f \geq 0$, then $m(f) \geq 0$, and
- (3) $m(\gamma f) = m(f)$ for each $\gamma \in G$ and $f \in L^\infty(G, \mathbb{R})$, where $(\gamma f)(\gamma') = f(\gamma^{-1}\gamma')$.

For instance, groups with subexponential growth and finite extensions of solvable groups are amenable whereas groups containing a free subgroup on two generators are not amenable. The proof of (2.6) given in [FM07] is based on an equivariant version of the weak KAM theorem and can be extended as follows. Let \widetilde{M} be a regular cover of M with amenable covering transformation group G . Let \widehat{M} denote a subcover whose covering transformation group \widehat{G} is isomorphic to the abelianization $G/[G, G]$. Note that \widehat{M} is covered by $\widehat{M}^{\text{abel}}$, which entails a surjective homomorphism $\Phi: H_1(M, \mathbb{Z}) \rightarrow \widehat{G}$, whose kernel is the group of covering transformations of $\widehat{M}^{\text{abel}}$ with trivial projections to \widehat{M} . The transpose of Φ can be extended to an injective linear map from $\text{Hom}(\widehat{G}, \mathbb{Z}) \otimes \mathbb{R} \simeq \text{Hom}(\widehat{G}, \mathbb{R}) \simeq \text{Hom}(G, \mathbb{R})$ into $\text{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z}) \otimes \mathbb{R} \simeq H^1(M, \mathbb{Z}) \otimes \mathbb{R} \simeq H^1(M, \mathbb{R})$, whereby $\text{Hom}(G, \mathbb{R})$ can be identified with a subspace of $H^1(M, \mathbb{R})$, namely,

$$(2.7) \quad \left\{ [\omega] \in H^1(M, \mathbb{R}) \mid \int_{\gamma} \widehat{\omega} = 0 \text{ for any closed curve } \gamma \text{ in } \widehat{M} \right\},$$

where $\widehat{\omega}$ denotes the lift of a representative of $[\omega]$. We provide an explicit isomorphism below. A similar argument appears in [KS00], which deals with the long-time asymptotics of the heat kernel. Note that any $\widehat{\omega} \in \Omega^1(\widehat{M}, \mathbb{R})$ that satisfies (2.7) is exact with primitives of the form $f(x) = f(x_0) + \int_{x_0}^x \widehat{\omega}$, where $x_0 \in \widehat{M}$ and $f(x_0) \in \mathbb{R}$ can be chosen arbitrarily. Hence, $[\omega] \in \text{Hom}(G, \mathbb{R})$ if and only if the lift of any representative of $[\omega]$ to \widehat{M} is exact. The following theorem generalizes the aforementioned results in [PP97, FM07].

Theorem 2. *Let M be a connected closed Riemannian manifold. Let \widetilde{M} be a regular cover with amenable covering transformation group G , and let \widehat{M} be a subcover whose covering transformation group \widehat{G} is isomorphic to $G/[G, G]$. Then, for any $\alpha \in \Omega^1(M, \mathbb{R})$ and $\mathbf{V} \in C^\infty(M, \mathbb{R})$, the Lagrangian $L_{\alpha, \mathbf{V}}: TM \rightarrow \mathbb{R}$ given by*

$$L_{\alpha, \mathbf{V}}(x, v) = \frac{1}{2}|v|_x^2 - \alpha_x(v) - \mathbf{V}(x)$$

and its lifts $L_{\widetilde{\alpha}, \widetilde{\mathbf{V}}}$ and $L_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ to \widetilde{M} and \widehat{M} satisfy

$$\mathbf{c}(L_{\widetilde{\alpha}, \widetilde{\mathbf{V}}}) = \mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{V}}}) = \min_{[\omega] \in \text{Hom}(G, \mathbb{R})} \mathbf{c}(L_{\alpha - \omega, \mathbf{V}}) = \min_{\omega \in \Omega^1(M, \mathbb{R}) : \widehat{\omega} \text{ is exact}} \mathbf{c}(L_{\alpha - \omega, \mathbf{V}}),$$

where $\text{Hom}(G, \mathbb{R})$ denotes the vector space (2.7), and $\widehat{\omega}$ denotes the lift of ω to \widehat{M} .

Note that forms with exact lifts are necessarily closed. For abelian covers $\widetilde{M} = \widehat{M}$, one obtains the following generalization of $\mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{v}}}^{\text{abel}}) = \mathbf{c}_0(L_{\alpha, \mathbf{v}})$.

Corollary 3. *If \widehat{M} is a regular cover with abelian covering transformation group, then we have*

$$\mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{v}}}) = \min_{\omega \in \Omega^1(M, \mathbb{R}) : \widehat{\omega} \text{ is exact}} \mathbf{c}(L_{\alpha - \omega, \mathbf{v}}).$$

Proof of Theorem 2. For any $\omega \in \Omega^1(M, \mathbb{R})$ with exact lift $\widehat{\omega} \in \Omega^1(\widehat{M}, \mathbb{R})$, (2.3) and (2.5) imply that

$$\mathbf{c}(L_{\widetilde{\alpha}, \widetilde{\mathbf{v}}}) \leq \mathbf{c}(L_{\widehat{\alpha}, \widehat{\mathbf{v}}}) = \mathbf{c}(L_{\widehat{\alpha} - \widehat{\omega}, \widehat{\mathbf{v}}}) \leq \mathbf{c}(L_{\alpha - \omega, \mathbf{v}}),$$

see also [PP97, Lemma 2.4]. Hence, Theorem 2 is proven once we find $\omega \in \Omega^1(M, \mathbb{R})$ with $[\omega] \in \text{Hom}(G, \mathbb{R})$ such that $\mathbf{c}(L_{\alpha - \omega, \mathbf{v}}) = \mathbf{c}(L_{\widetilde{\alpha}, \widetilde{\mathbf{v}}})$. This is established along the lines of [FM07, Theorem 1.5], more precisely, \widetilde{M} , \widehat{M} , G and \widehat{G} assume the roles of $\widehat{M}^{\text{univ}}$, $\widehat{M}^{\text{abel}}$, $\pi_1(M)$ and $H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$, respectively. For any ω with $[\omega] \in \text{Hom}(G, \mathbb{R})$, we can choose $f_\omega \in C^\infty(\widehat{M}, \mathbb{R})$ such that the lift of ω to \widehat{M} takes the form $\widetilde{\omega} = df_\omega$. Since \widehat{M} is connected, f_ω is determined up to a constant. For any $\gamma \in G$, we consider the function $\rho_{\omega, \gamma} = \gamma^* f_\omega - f_\omega$. Note that the definition of $\rho_{\omega, \gamma}$ is independent of the choice of f_ω . Since $\gamma^* \widetilde{\omega} = \widetilde{\omega}$ for any $\gamma \in G$, the function $\rho_{\omega, \gamma}$ has vanishing derivative and is therefore a constant that we denote by $\rho_\omega(\gamma)$. One easily verifies that the mapping $\gamma \mapsto \rho_\omega(\gamma)$ is an element of $\text{Hom}(G, \mathbb{R})$. Moreover, the mapping $\omega \mapsto \rho_\omega$ is linear and injective, and thus establishes the desired isomorphism. In [FM07, Section 7], it is shown that any $\rho \in \text{Hom}(G, \mathbb{R})$ gives rise to a critical value $\mathbf{c}(\rho)$ such that any ρ_ω as above satisfies $\mathbf{c}(\rho_\omega) = \mathbf{c}(L_{\alpha + \omega, \mathbf{v}})$. Since G is amenable, there exists $\rho \in \text{Hom}(G, \mathbb{R})$ such that $\mathbf{c}(\rho) = \mathbf{c}(L_{\widetilde{\alpha}, \widetilde{\mathbf{v}}})$ by virtue of [FM07, Lemma 7.3 and Theorem 7.4]. We already saw that $\rho = \rho_\omega$ for some ω with $[\omega] \in \text{Hom}(G, \mathbb{R})$, which completes the proof. \square

Note that the functions $f_\omega \in C^\infty(\widehat{M}, \mathbb{R})$ in the proof of Theorem 2 are lifts of smooth functions on the subcover \widehat{M} . Algebraically, this is reflected in $\text{Hom}(G, \mathbb{R}) \simeq \text{Hom}(\widehat{G}, \mathbb{R})$. The existence of $\rho_\omega \in \text{Hom}(G, \mathbb{R})$ with minimal associated critical value $\mathbf{c}(\rho_\omega)$ is a consequence of the convexity and superlinearity of the Mather function [FM07, Proposition 7.2] which is the mapping $\text{Hom}(G, \mathbb{R}) \ni \rho_\omega \mapsto \mathbf{c}(\rho_\omega) = \mathbf{c}(L_{\alpha + \omega, \mathbf{v}})$. One easily extends Theorem 2 to any Lagrangian $L: TM \rightarrow \mathbb{R}$ of class C^2 that satisfies the following conditions:

- (1) Convexity: For every $x \in M$, the restriction of L to $T_x M$ has positive definite Hessian everywhere.
- (2) Uniform superlinearity: For every $K \geq 0$, there exists $C(K) \in \mathbb{R}$ such that

$$L(x, v) \geq K |v|_x - C(K) \quad \text{for all } (x, v) \in TM.$$

Corollary 3 also allows for a direct proof along the lines of [PP97, Theorem 1.1]. More precisely, if G denotes the abelian covering transformation group of \widehat{M} , then \widehat{M} , G , $G \otimes \mathbb{R}$ and $\text{Hom}(G, \mathbb{R})$ assume the roles of $\widehat{M}^{\text{abel}}$, $H_1(M, \mathbb{Z})$, $H_1(M, \mathbb{R})$ and $H^1(M, \mathbb{R})$ in [PP97], respectively. In particular, the curves \overline{x}_i appearing in the proof of [PP97, Theorem 1.1] can be chosen as lifts of curves in \widehat{M} , for which reason the slopes appearing in [PP97, Theorem 2.3] can be taken from $\text{Hom}(G, \mathbb{R})$. The necessary adaptations are routine but lengthy for which reason we skip them.

3. MAGNETIC SCHRÖDINGER OPERATORS

As in Section 2, let M be a connected closed manifold with Riemannian metric g , and let \widehat{M} be a regular cover of M equipped with potentials $\widehat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$ and $\widehat{V} \in C^\infty(\widehat{M}, \mathbb{R})$. Recall that the system $(T^*\widehat{M}, \widehat{\omega}_0, H_{\widehat{\alpha}, \widehat{V}})$ with standard symplectic structure $\widehat{\omega}_0$ on $T^*\widehat{M}$ and Hamiltonian $H_{\widehat{\alpha}, \widehat{V}}: T^*\widehat{M} \rightarrow \mathbb{R}$ given by

$$(3.1) \quad H_{\widehat{\alpha}, \widehat{V}}(x, p) = \frac{1}{2} |p + \widehat{\alpha}|_x^2 + \widehat{V}(x)$$

describes the classical motion of a charged particle on \widehat{M} under the influence of the electromagnetic field with magnetic and electric potentials $\widehat{\alpha}$ and \widehat{V} , respectively. In contrast, non-relativistic quantum mechanics is essentially the study of self-adjoint, densely-defined operators on Hilbert spaces. In our case, these are minimal Schrödinger operators, that is, closures of differential operators of Schrödinger type that are initially defined on the space $C_0^\infty(\widehat{M}) = C_0^\infty(\widehat{M}, \mathbb{C})$ of compactly-supported, smooth, \mathbb{C} -valued functions on \widehat{M} . In comparison to Laplacians, the study of magnetic potentials requires to consider complex-valued functions. Let $L^2(\widehat{M}) = L^2(\widehat{M}, d\text{vol}, \mathbb{C})$ denote the completion of $C_0^\infty(\widehat{M})$ with respect to the norm $\|u\| = \sqrt{\langle u, u \rangle}$ coming from the inner product

$$\langle u, v \rangle = \int_{\widehat{M}} u \bar{v} d\text{vol},$$

where $u, v \in C_0^\infty(\widehat{M})$ and $d\text{vol}$ denotes the volume form of g . Following [Pat01], we use the Dirac quantization rule which says that in order to quantize (3.1), we have to replace p by the operator $\frac{1}{i}d$, where d is the exterior differential. Let $\Omega_0^1(\widehat{M}) = \Omega_0^1(\widehat{M}, \mathbb{C})$ denote the space of compactly-supported, smooth, \mathbb{C} -valued 1-forms on \widehat{M} . This space is equipped with an inner product given by integration over the fibrewise inner products on $T^*\widehat{M}$. We denote the completion of $\Omega_0^1(\widehat{M})$ by $L^2(\Omega^1(\widehat{M}))$. The magnetic differential $d_{\widehat{\alpha}}: C_0^\infty(\widehat{M}) \rightarrow \Omega_0^1(\widehat{M})$ is the operator given by

$$d_{\widehat{\alpha}}u = \frac{1}{i} du + u \widehat{\alpha}.$$

The associated magnetic Schrödinger operator $\mathcal{H}_{\widehat{\alpha}, \widehat{V}}$ with domain $C_0^\infty(\widehat{M})$ is defined as

$$(3.2) \quad \mathcal{H}_{\widehat{\alpha}, \widehat{V}} = \frac{1}{2} d_{\widehat{\alpha}}^* d_{\widehat{\alpha}} + \widehat{V},$$

where $d_{\widehat{\alpha}}^*: \Omega_0^1(\widehat{M}) \rightarrow C_0^\infty(\widehat{M})$ denotes the formal adjoint of $d_{\widehat{\alpha}}$. Recall that $d_{\widehat{\alpha}}^*$ is the unique differential operator such that $\langle u, d_{\widehat{\alpha}}^* \omega \rangle = \langle d_{\widehat{\alpha}} u, \omega \rangle$ holds for any $u \in C_0^\infty(\widehat{M})$ and $\omega \in \Omega_0^1(\widehat{M})$. As a differential operator, $\mathcal{H}_{\widehat{\alpha}, \widehat{V}}$ can be expressed in local terms.

Lemma 4. *The magnetic Schrödinger operator (3.2) with domain $C_0^\infty(\widehat{M})$ is given by*

$$(3.3) \quad \mathcal{H}_{\widehat{\alpha}, \widehat{V}}u = \frac{1}{2} \Delta u - i \langle du, \widehat{\alpha} \rangle + \left(\frac{1}{2} d^* \widehat{\alpha} + \frac{1}{2} |\widehat{\alpha}|^2 + \widehat{V} \right) u,$$

where d^* denotes the codifferential and $\Delta = d^*d$ is the Laplace-Beltrami operator on \widehat{M} . With respect to local coordinates (x_1, x_2, \dots, x_n) , we have

$$(3.4) \quad \mathcal{H}_{\widehat{\alpha}, \widehat{V}} = \frac{1}{2} \frac{1}{\sqrt{|g|}} \sum_{j,k} \left(\frac{1}{i} \frac{\partial}{\partial x_j} + \overline{\widehat{\alpha}}_j \right) g^{jk} \sqrt{|g|} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + \widehat{\alpha}_k \right) + \widehat{V},$$

where $\widehat{\alpha}(x_1, x_2, \dots, x_n) = \sum_k \widehat{\alpha}_k(x_1, x_2, \dots, x_n) dx^k$, $dvol = \sqrt{|g|} dx^1 \dots dx^n$ and $g^{jk} = \langle dx^j, dx^k \rangle$ are the entries of the inverse of the local matrix expression of g .

Proof. For compact \widehat{M} , [Pat01] contains a proof of (3.3) that can be easily generalized to our setting. Instead, we verify the coordinate version (3.4). For any $u \in C_0^\infty(\widehat{M})$ with support in the given coordinate neighborhood and any $\omega \in \Omega_0^1(\widehat{M})$ with local expression $\omega = \sum_k \omega_k dx^k$, partial integration yields

$$\begin{aligned} \langle u, d_{\widehat{\alpha}}^* \omega \rangle &= \langle -i du + u \widehat{\alpha}, \omega \rangle = \left\langle \sum_j \left(\frac{1}{i} \frac{\partial u}{\partial x_j} + u \widehat{\alpha}_j \right) dx^j, \sum_k \omega_k dx^k \right\rangle \\ &= \int \sum_{j,k} \left(\left(\frac{1}{i} \frac{\partial u}{\partial x_j} + u \widehat{\alpha}_j \right) \overline{\omega}_k g^{jk} \sqrt{|g|} \right) dx^1 \dots dx^n \\ &= \int u \sum_{j,k} \left(\left(-\frac{1}{i} \frac{\partial}{\partial x_j} + \widehat{\alpha}_j \right) g^{jk} \sqrt{|g|} \overline{\omega}_k \right) dx^1 \dots dx^n \\ &= \left\langle u, \frac{1}{\sqrt{|g|}} \sum_{j,k} \left(\frac{1}{i} \frac{\partial}{\partial x_j} + \overline{\widehat{\alpha}}_j \right) g^{jk} \sqrt{|g|} \omega_k \right\rangle. \end{aligned}$$

If $\omega = d_{\widehat{\alpha}} v$ for some $v \in C_0^\infty(\widehat{M})$, then $\omega_k = \left(\frac{1}{i} \frac{\partial}{\partial x_k} + \widehat{\alpha}_k \right) v$, which completes the proof. \square

Theorem 5. *If \widehat{V} is semi-bounded from below, meaning $\inf \widehat{V} > -\infty$, then $\mathcal{H}_{\widehat{\alpha}, \widehat{V}}$ with domain $C_0^\infty(\widehat{M})$ is essentially self-adjoint, that is, its closure $\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}}$ is self-adjoint.*

Proof. As a cover of a closed manifold, \widehat{M} is complete. Moreover, $\mathcal{H}_{\widehat{\alpha}, \widehat{V}}$ is semi-bounded from below on $C_0^\infty(\widehat{M})$ since for any $u \in C_0^\infty(\widehat{M})$, we have

$$(3.5) \quad \langle \mathcal{H}_{\widehat{\alpha}, \widehat{V}} u, u \rangle = \frac{1}{2} \langle d_{\widehat{\alpha}} u, d_{\widehat{\alpha}} u \rangle + \langle u \widehat{V}, u \rangle \geq \inf \widehat{V} \langle u, u \rangle.$$

The claim now follows from [Shu01, Theorem 1.1] or [BMS02, Theorem 2.13]. \square

Ikeba and Kato [IK62] were the first to prove essential self-adjointness for a wide class of singular magnetic potentials on \mathbb{R}^n . Kato [Kat72] extended these results using his famous inequality, which we discuss in Section 3.1. Hess et al. [HSU77] and Simon [Sim79] later revealed the functional analytic nature of Kato's inequality in terms of domination of semi-groups. Most proofs of essential self-adjointness of magnetic Schrödinger operators use some sort of Kato inequality, see also [LS81, Iwa90, BMS02, Mil03, Mil04, RM05].

The study of $\mathcal{H}_{\widehat{\alpha}, 0}$ dates back to the 1930s when Landau considered the case $\widehat{M} = \mathbb{R}^2$; however, major progress had not been made till the 1970s [AHS78, Iwa86, Tam87, Tam88, Tam89]. In this context, we point out the article [KS02], in which Kondratiev and Shubin derive necessary and sufficient conditions for magnetic Schrödinger operators to have discrete

spectrum. In particular, they recover a theorem of Avron et al. [AHS78] which says that whenever $\overline{\mathcal{H}_{0,0}} = \overline{\Delta}$ has discrete spectrum, the same is true for any $\overline{\mathcal{H}_{\hat{\alpha},0}}$ with $\hat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$.

By virtue of [Dav95, Theorem 4.3.1], the estimate (3.5) implies $\text{spec}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}) \subseteq [\inf \hat{V}, \infty)$. Moreover, $\mathcal{H}_{\hat{\alpha},\hat{V}}$ and $\mathcal{H}_{-\hat{\alpha},\hat{V}}$ are conjugate to each other with respect to complex conjugation on $C_0^\infty(\widehat{M})$ which yields $\text{spec}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}) = \text{spec}(\overline{\mathcal{H}_{-\hat{\alpha},\hat{V}}})$. If \widehat{M} is compact, then $\text{spec}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}})$ is discrete and $L^2(\widehat{M})$ is the direct sum of countably many finite-dimensional eigenspaces of $\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}$ consisting of smooth eigenfunctions as is well-known, see [Shi87, Theorem 2.1] for instance. If, in addition, \widetilde{M} is a finite regular cover of \widehat{M} with lifted potentials $\tilde{\alpha}$ and \tilde{V} , then we obtain $\text{spec}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}) \subseteq \text{spec}(\overline{\mathcal{H}_{\tilde{\alpha},\tilde{V}}})$ by lifting eigenfunctions.

In the following, we collect results about magnetic Schrödinger operators that describe periodic electromagnetic fields. Let G denote the covering transformation group of the regular cover $\hat{\pi}: \widehat{M} \rightarrow M$. We assume that $d\hat{\alpha} = \hat{\pi}^*\beta$ for some magnetic field $\beta \in \Omega^2(M, \mathbb{R})$, and that $\hat{V} = V \circ \hat{\pi}$ for some electric potential $V \in C^\infty(M, \mathbb{R})$. As before, $[\beta] = 0 \in H^2(M, \mathbb{R})$ is called the exact case, whereas $[\beta] \neq 0 \in H^2(M, \mathbb{R})$ is called the monopole case. For any $\gamma \in G$, the 1-form $\gamma^*\hat{\alpha} - \hat{\alpha}$ is closed since $d(\gamma^*\hat{\alpha}) = \gamma^*\hat{\pi}^*\beta = \hat{\pi}^*\beta = d\hat{\alpha}$, and we define the $\hat{\alpha}$ -exact subgroup as

$$G^{\hat{\alpha}} = \{\gamma \in G \mid \gamma^*\hat{\alpha} - \hat{\alpha} \text{ is exact}\}.$$

Note that if $\widehat{M} = \widehat{M}^{\text{univ}}$, we have $G^{\hat{\alpha}} = G = \pi_1(M)$. The subgroup property can be seen as follows. If $\gamma, \gamma' \in G^{\hat{\alpha}}$ such that $\gamma^*\hat{\alpha} = \hat{\alpha} + df_\gamma$ and $\gamma'^*\hat{\alpha} = \hat{\alpha} + df_{\gamma'}$ where $f_\gamma, f_{\gamma'} \in C^\infty(\widehat{M}, \mathbb{R})$, then

$$(\gamma'\gamma)^*\hat{\alpha} = \gamma^*(\hat{\alpha} + df_{\gamma'}) = \hat{\alpha} + d(f_\gamma + \gamma^*f_{\gamma'}).$$

Theorem 6. *If $G^{\hat{\alpha}}$ is infinite, then all eigenspaces of $\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}$ are infinite-dimensional, that is, $\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}$ has no discrete spectrum.*

The idea behind Theorem 6 goes back to the following well-known argument for the special case $\hat{\alpha} = 0$ and $\hat{V} = 0$. Since Laplacians commute with isometries, any eigenfunction can be translated by covering transformations to obtain linearly independent eigenfunctions in the same eigenspace. In the presence of magnetic potentials, one has to adapt the translations as follows. If $x_0 \in \widehat{M}$ denotes some fixed reference point, then any $\gamma \in G^{\hat{\alpha}}$ gives rise to a unique $f_\gamma \in C^\infty(\widehat{M}, \mathbb{R})$ such that $\gamma^*\hat{\alpha} = \hat{\alpha} + df_\gamma$ and $f_\gamma(x_0) = 0$. Following [MS02], we define the associated magnetic translation $T_\gamma: L^2(\widehat{M}) \rightarrow L^2(\widehat{M})$ as the unitary map given by $T_\gamma u = e^{if_\gamma} \gamma^* u$.

Lemma 7. *Magnetic translations map $\text{Dom}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}})$ into itself and commute with $\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}$, that is, for any $\gamma \in G^{\hat{\alpha}}$ and $u \in \text{Dom}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}})$, we have $e^{if_\gamma} \gamma^* u \in \text{Dom}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}})$ and*

$$(3.6) \quad \overline{\mathcal{H}_{\hat{\alpha},\hat{V}}} (e^{if_\gamma} \gamma^* u) = e^{if_\gamma} \gamma^* (\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}} u).$$

Proof. We use ideas from [MS02], where the underlying twisted group algebra structure is exploited in detail. In order to see that (3.6) holds for any $u \in \text{Dom}(\overline{\mathcal{H}_{\hat{\alpha},\hat{V}}}) = C_0^\infty(\widehat{M})$, it suffices to show that for any $v \in C_0^\infty(\widehat{M})$

$$(3.7) \quad \langle \overline{\mathcal{H}_{\hat{\alpha},\hat{V}}} T_\gamma u, v \rangle = \langle T_\gamma \overline{\mathcal{H}_{\hat{\alpha},\hat{V}}} u, v \rangle.$$

One can prove (3.7) directly using (3.3). In order to avoid this tedious calculation, we define a magnetic translation T_γ on $\Omega_0^1(\widehat{M})$ given as $T_\gamma \omega = e^{i f_\gamma} \gamma^* \omega$ for $\omega \in \Omega_0^1(\widehat{M})$. We obtain $d_{\widehat{\alpha}} \circ T_\gamma = T_\gamma \circ d_{\widehat{\alpha}}$ by noting that for any $u \in C_0^\infty(\widehat{M})$,

$$\begin{aligned} d_{\widehat{\alpha}} T_\gamma u &= -i d(e^{i f_\gamma} \gamma^* u) + e^{i f_\gamma} \gamma^* u \widehat{\alpha} \\ &= e^{i f_\gamma} (-i \gamma^* du + \gamma^* u (df_\gamma + \widehat{\alpha})) = e^{i f_\gamma} \gamma^* (-i du + u \widehat{\alpha}) = T_\gamma d_{\widehat{\alpha}} u, \end{aligned}$$

where we used that $df_\gamma + \widehat{\alpha} = \gamma^* \widehat{\alpha}$. Taking adjoints, we obtain $T_\gamma^{-1} \circ d_{\widehat{\alpha}}^* = d_{\widehat{\alpha}}^* \circ T_\gamma^{-1}$ on $\Omega_0^1(\widehat{M})$. Since $T_\gamma^{-1} \circ \widehat{V} \circ T_\gamma = \widehat{V}$ trivially holds on $C_0^\infty(\widehat{M})$, the claim $\mathcal{H}_{\widehat{\alpha}, \widehat{V}} T_\gamma = T_\gamma \mathcal{H}_{\widehat{\alpha}, \widehat{V}}$ follows directly from (3.2). Let now $u \in \text{Dom}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}})$ and choose a sequence $(u_n)_{n \in \mathbb{N}}$ in $C_0^\infty(\widehat{M}) = \text{Dom}(\mathcal{H}_{\widehat{\alpha}, \widehat{V}})$ such that $\|u - u_n\| \rightarrow 0$ and $\|\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}} u - \mathcal{H}_{\widehat{\alpha}, \widehat{V}} u_n\| \rightarrow 0$. The translated sequence $(T_\gamma u_n)_{n \in \mathbb{N}}$ also lies in $C_0^\infty(\widehat{M})$ and satisfies $\|T_\gamma u - T_\gamma u_n\| = \|u - u_n\| \rightarrow 0$ as well as

$$\left\| T_\gamma \overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}} u - \mathcal{H}_{\widehat{\alpha}, \widehat{V}} T_\gamma u_n \right\| = \left\| T_\gamma \overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}} u - T_\gamma \mathcal{H}_{\widehat{\alpha}, \widehat{V}} u_n \right\| \rightarrow 0$$

since T_γ is unitary. Hence, $T_\gamma u \in \text{Dom}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}})$ and $\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}} T_\gamma u = T_\gamma \overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}} u$ as claimed. \square

Proof of Theorem 6. We largely follow [Sun88] and assume that $\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}}$ has an eigenspace with finite orthonormal basis $\{u_1, u_2, \dots, u_N\}$. For each $\gamma \in G^{\widehat{\alpha}}$ and $j \in \{1, 2, \dots, N\}$, the translate $T_\gamma u_j$ is an eigenfunction of $\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}}$ by virtue of Lemma 7. Hence,

$$T_\gamma u_j = \sum_{k=1}^N U_{jk}(\gamma) u_k$$

for some matrix $U(\gamma)$, which is unitary since

$$\sum_{k=1}^N U_{jk}(\gamma) \overline{U_{lk}(\gamma)} = \sum_{k,m=1}^N U_{jk}(\gamma) \overline{U_{lm}(\gamma)} \langle u_k, u_m \rangle = \langle T_\gamma u_j, T_\gamma u_l \rangle = \langle u_j, u_l \rangle.$$

Let F denote a fundamental domain for the action of $G^{\widehat{\alpha}}$ on \widehat{M} . We obtain

$$\begin{aligned} N = \sum_{j=1}^N \|u_j\|^2 &= \sum_{j=1}^N \sum_{\gamma \in G^{\widehat{\alpha}}} \int_F |\gamma^* u_j|^2 = \sum_{\gamma \in G^{\widehat{\alpha}}} \sum_{j=1}^N \int_F |T_\gamma u_j|^2 \\ &= \sum_{\gamma \in G^{\widehat{\alpha}}} \sum_{j=1}^N \sum_{k,l=1}^N \int_F U_{jk}(\gamma) u_k \overline{U_{jl}(\gamma)} \overline{u_l} = |G^{\widehat{\alpha}}| \sum_{k=1}^N \int_F |u_k|^2, \end{aligned}$$

which contradicts $|G^{\widehat{\alpha}}| = \infty$. \square

3.1. Ground state energy. Let $\widehat{\pi}: \widehat{M} \rightarrow M$ be a regular cover equipped with potentials $\widehat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$ and $\widehat{V} = V \circ \widehat{\pi}$, where $V \in C^\infty(M, \mathbb{R})$.

Definition 8. The ground state energy is defined as

$$\lambda_0(\widehat{\alpha}, \widehat{V}) = \lambda_0(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}}) = \inf \text{spec}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}}).$$

Any normalized eigenfunction of $\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}}$ with eigenvalue $\lambda_0(\widehat{\alpha}, \widehat{V})$ is called a ground state.

If \widehat{M} is non-compact, then $\lambda_0(\widehat{\alpha}, \widehat{V})$ can belong to the continuous spectrum $\text{spec}_{\text{cont}}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}})$. In any case, the variational principle says that $\lambda_0(\widehat{\alpha}, \widehat{V})$ is an infimum of Rayleigh quotients

$$\lambda_0(\widehat{\alpha}, \widehat{V}) = \inf_{u \in C_0^\infty(\widehat{M}) \setminus \{0\}} \frac{\langle \mathcal{H}_{\widehat{\alpha}, \widehat{V}} u, u \rangle}{\langle u, u \rangle} = \inf_{u \in C_0^\infty(\widehat{M}) \setminus \{0\}} \frac{\int_{\widehat{M}} \frac{1}{2} |d_{\widehat{\alpha}} u|^2 + \widehat{V} |u|^2}{\int_{\widehat{M}} |u|^2} \geq \min \mathbf{V}.$$

The reader is referred to [Dav95, Section 4] for details. If $\mathbf{V} = 0$, we use the shortened notation $\lambda_0(\widehat{\alpha})$ for $\lambda_0(\widehat{\alpha}, 0)$. We recall that a distribution ν on \widehat{M} is called positive, denoted by $\nu \geq 0$, if $\nu(f) \geq 0$ for every non-negative $f \in C_0^\infty(\widehat{M}, \mathbb{R})$. The following version of Kato's inequality is a special case of [BMS02, Proposition 5.9 and Corollary 5.10], see also [HSU80, Proposition 2.2].

Proposition 9. *For any $u \in C_0^\infty(\widehat{M})$ and $\varepsilon > 0$, let $|u|_\varepsilon \in C_0^\infty(\widehat{M})$ be defined as*

$$|u|_\varepsilon = \sqrt{|u|^2 + \varepsilon^2} - \varepsilon.$$

Then, we have the following inequality of smooth functions on \widehat{M}

$$(|u|_\varepsilon + \varepsilon) \mathcal{H}_{0,0} |u|_\varepsilon \leq \text{Re} \langle \mathcal{H}_{\widehat{\alpha}, 0} u, u \rangle.$$

In the limit $\varepsilon \searrow 0$, we moreover have the following inequality of distributions

$$\mathcal{H}_{0,0} |u| \leq \text{Re} \langle \mathcal{H}_{\widehat{\alpha}, 0} u, \text{sign } u \rangle,$$

where

$$\text{sign } u = \begin{cases} \frac{u(x)}{|u(x)|} & \text{for } u(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For compact \widehat{M} , let $H^1(\widehat{M}, \mathbb{Z})$ denote the first cohomology group with integer coefficients, which we identify with the lattice group

$$\left\{ [\omega] \in H^1(\widehat{M}, \mathbb{R}) \mid \int_\gamma \omega \in \mathbb{Z} \text{ for any closed curve } \gamma \text{ in } \widehat{M} \right\}.$$

The following theorem describes the diamagnetic effect of magnetic fields and the gauge invariance group of compact covers.

Theorem 10. *For any regular cover $\widehat{\pi}: \widehat{M} \rightarrow M$ equipped with potentials $\widehat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$ and $\widehat{V} = \mathbf{V} \circ \widehat{\pi}$, where $\mathbf{V} \in C^\infty(M, \mathbb{R})$, the diamagnetic inequality holds, namely*

$$(3.8) \quad \lambda_0(0, \widehat{V}) \leq \lambda_0(\widehat{\alpha}, \widehat{V}).$$

Moreover, any $\omega = \frac{1}{i} \frac{d\varphi}{\varphi}$ with $\varphi \in C^\infty(\widehat{M}, \mathbb{S}^1) \subset C^\infty(\widehat{M})$ gives rise to a unitary gauge transformation $\mathcal{U}_\varphi: L^2(\widehat{M}) \rightarrow L^2(\widehat{M})$ given by $\mathcal{U}_\varphi u = \varphi u$ such that

$$(3.9) \quad \mathcal{H}_{\widehat{\alpha} + \omega, \widehat{V}} = \mathcal{U}_\varphi^* \mathcal{H}_{\widehat{\alpha}, \widehat{V}} \mathcal{U}_\varphi \quad \text{on } C_0^\infty(\widehat{M}).$$

In particular,

$$(3.10) \quad \lambda_0(\widehat{\alpha} + df, \widehat{V}) = \lambda_0(\widehat{\alpha}, \widehat{V}) \quad \text{for } f \in C^\infty(\widehat{M}, \mathbb{R}).$$

If \widehat{M} is a finite and therefore compact regular cover of M , then the following are equivalent:

$$(1) \quad \lambda_0(\widehat{\alpha}, \widehat{V}) = \lambda_0(0, \widehat{V}),$$

(2) $\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}$ and $\overline{\mathcal{H}_{0, \hat{V}}}$ are unitarily equivalent via a gauge transformation, in particular,

$$\text{spec}(\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}) = \text{spec}(\overline{\mathcal{H}_{0, \hat{V}}}),$$

(3) $\hat{\alpha} = \frac{1}{i} \frac{d\varphi}{\varphi}$ for some $\varphi \in C^\infty(\widehat{M}, \mathbb{S}^1) \subset C^\infty(\widehat{M})$,

(4) $\hat{\alpha}$ is closed and $[\hat{\alpha}] \in 2\pi H^1(\widehat{M}, \mathbb{Z})$, that is, for any closed curve γ in \widehat{M}

$$\int_\gamma \hat{\alpha} \in 2\pi \mathbb{Z}.$$

Proof. The lifted potential \widehat{V} is bounded from below by $\min \mathbf{V}$, which exists by compactness of M . Hence, we may assume that $\widehat{V} \geq 0$, otherwise consider $\mathcal{H}_{\hat{\alpha}, \hat{V}} - \min \mathbf{V}$. In order to prove inequality (3.8), one can argue along the lines of [HSU77, Theorem 3.3] and use Kato's inequality as stated in Proposition 9 to verify that the semigroup $e^{-t\overline{\mathcal{H}_{0, \hat{V}}}}$ dominates the semigroup $e^{-t\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}}$, which implies (3.8) by virtue of [HSU77, Corollary 2.13]. The gauge equivalence (3.9) is the content of [Shi87, Proposition 3.2], where the local expression (3.3) is used. In order to provide an alternative proof, we let $\mathcal{W}_\varphi: L^2(\Omega^1(\widehat{M})) \rightarrow L^2(\Omega^1(\widehat{M}))$ denote the unitary map given by $\mathcal{W}_\varphi \omega = \varphi \omega$. Note that $\mathcal{U}_\varphi(C_0^\infty(\widehat{M})) = C_0^\infty(\widehat{M})$ and $\mathcal{U}_\varphi^* = \mathcal{U}_\varphi^{-1} = \mathcal{U}_{\varphi^{-1}}$ as well as $\mathcal{W}_\varphi(\Omega_0^1(\widehat{M})) = \Omega_0^1(\widehat{M})$ and $\mathcal{W}_\varphi^* = \mathcal{W}_\varphi^{-1} = \mathcal{W}_{\varphi^{-1}}$. For any $u \in C_0^\infty(\widehat{M})$, we have

$$d_{\hat{\alpha}} \mathcal{U}_\varphi u = d_{\hat{\alpha}}(\varphi u) = -i \varphi du + u(-i d\varphi + \varphi \hat{\alpha}) = \varphi d_{\hat{\alpha}+\omega} u = \mathcal{W}_\varphi d_{\hat{\alpha}+\omega} u.$$

Taking adjoints, we get $\mathcal{U}_\varphi^* d_{\hat{\alpha}}^* = d_{\hat{\alpha}+\omega}^* \mathcal{W}_\varphi^*$ on $\Omega_0^1(\widehat{M})$. In addition, $\mathcal{U}_\varphi^* \widehat{V} \mathcal{U}_\varphi = \widehat{V}$ trivially holds on $C_0^\infty(\widehat{M})$. Hence, we get (3.9), which in turn implies (3.10) by setting $\varphi = e^{if}$.

In what follows, let \widehat{M} be compact. The equivalence of 3. and 4. is the content of [Shi87, Proposition 3.1]. Since 3. implies 2. by the first part, and since 2. immediately gives 1., it suffices to show that 1. implies 3. We use ideas from [Hel88]. It is well-known that $\overline{\mathcal{H}_{0, \hat{V}}}$ has a smooth positive ground state u_0 , in particular, for any $u \in C^\infty(\widehat{M})$

$$(3.11) \quad \lambda_0(0, \widehat{V}) \langle u_0^{-1} |u|^2, u_0 \rangle = \langle u_0^{-1} |u|^2, \mathcal{H}_{0, \hat{V}} u_0 \rangle = \frac{1}{2} \langle d(u_0^{-1} |u|^2), du_0 \rangle + \langle u_0^{-1} |u|^2 \widehat{V}, u_0 \rangle.$$

This yields another proof of the diamagnetic inequality (3.8) as follows

$$\begin{aligned} \|u_0 d_{\hat{\alpha}}(u_0^{-1} u)\|^2 &= \|i u_0^{-1} u du_0 - idu + u \hat{\alpha}\|^2 \\ &= \|-idu + u \hat{\alpha}\|^2 + 2 \text{Re} \langle -idu + u \hat{\alpha}, i u_0^{-1} u du_0 \rangle + \|u_0^{-1} u du_0\|^2 \\ &= \|d_{\hat{\alpha}} u\|^2 - 2 \text{Re} \langle u_0^{-1} \bar{u} du, du_0 \rangle + \langle u_0^{-2} |u|^2 du_0, du_0 \rangle \\ &= \|d_{\hat{\alpha}} u\|^2 - \langle d(u_0^{-1} |u|^2), du_0 \rangle \\ &= \|d_{\hat{\alpha}} u\|^2 + 2 \langle u_0^{-1} |u|^2 \widehat{V}, u_0 \rangle - 2 \lambda_0(0, \widehat{V}) \langle u_0^{-1} |u|^2, u_0 \rangle, \\ (3.12) \quad &= 2 \left\langle \left(\mathcal{H}_{\hat{\alpha}, \hat{V}} - \lambda_0(0, \widehat{V}) \right) u, u \right\rangle, \end{aligned}$$

where we used $\text{Re} \langle |u|^2 \hat{\alpha}, i u_0^{-1} du_0 \rangle = 0$ and (3.11). Recall that $\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}$ has a smooth normalized ground state [Shi87, Theorem 2.1], which we denote by $u_{\hat{\alpha}}$. Assuming that $\lambda_0(0, \widehat{V}) = \lambda_0(\hat{\alpha}, \widehat{V})$, we show that $u_{\hat{\alpha}}$ is non-vanishing. Due to (3.12), the function $\varphi_{\hat{\alpha}} = u_0^{-1} u_{\hat{\alpha}}$ satisfies $d_{\hat{\alpha}} \varphi_{\hat{\alpha}} = 0$, that is,

$$(3.13) \quad d\varphi_{\hat{\alpha}} = -i \varphi_{\hat{\alpha}} \hat{\alpha}.$$

We show that whenever $\varphi_{\hat{\alpha}}(x_0) = 0$ for some $x_0 \in \widehat{M}$, then $\varphi_{\hat{\alpha}}$ vanishes in a neighborhood of x_0 , which by continuity of $\varphi_{\hat{\alpha}}$ and connectivity of \widehat{M} leads to $\varphi_{\hat{\alpha}} = 0$ contradicting $\|u_{\hat{\alpha}}\| = 1$. Using (3.13), we can find $r_0 > 0$ and $C > 0$ such that the exponential map $\exp_{x_0}: T_{x_0}\widehat{M} \rightarrow \widehat{M}$ restricted to the ball $B(0, r_0)$ or radius r_0 is a diffeomorphism onto its image $B(x_0, r_0) = \exp_{x_0}(B(0, r_0))$, on which

$$|d\varphi_{\hat{\alpha}}| \leq C |\varphi_{\hat{\alpha}}|$$

holds pointwise. For $x \in B(x_0, r_0)$, let $v_x = \exp_{x_0}^{-1} x$, and note that since $\varphi_{\hat{\alpha}}$ is smooth,

$$\varphi_{\hat{\alpha}}(x) = \varphi_{\hat{\alpha}}(x) - \varphi_{\hat{\alpha}}(x_0) = \int_0^1 \frac{d}{dt} (\varphi_{\hat{\alpha}}(\exp_{x_0}(t v_x))) dt.$$

Using the Gauß lemma, we obtain that for any $0 < r < r_0$

$$\sup_{x \in B(x_0, r)} |\varphi_{\hat{\alpha}}(x)| \leq C r \sup_{x \in B(x_0, r)} |\varphi_{\hat{\alpha}}(x)|,$$

and $\varphi_{\hat{\alpha}}$ must vanish on any $B(x_0, r)$ with $0 < r < \min(r_0, C)$. Hence, $\varphi_{\hat{\alpha}}$ vanishes nowhere, and on any simply-connected coordinate neighborhood $U \subseteq \widehat{M}$, we may write $\varphi_{\hat{\alpha}} = R e^{if}$ for some $R \in C^\infty(U, \mathbb{R}^+)$ and $f \in C^\infty(U, \mathbb{R})$. Using (3.13), we obtain

$$\hat{\alpha} = i \varphi_{\hat{\alpha}}^{-1} d\varphi_{\hat{\alpha}} = i R^{-1} dR - df.$$

Since $\hat{\alpha}$ is real-valued, we have $\hat{\alpha} = -df$ and $dR = 0$, that is, $\hat{\alpha}$ is closed and $|\varphi_{\hat{\alpha}}|$ is constant on \widehat{M} . Hence, $\hat{\alpha} = \frac{1}{i} \frac{d\varphi}{\varphi}$ for $\varphi = \frac{\varphi_{\hat{\alpha}}}{|\varphi_{\hat{\alpha}}|} \in C^\infty(\widehat{M}, \mathbb{S}^1)$ as claimed. \square

Note that we can have $\lambda_0(\hat{\alpha}, \hat{V}) \neq \lambda_0(0, \hat{V})$ even if $d\hat{\alpha} = 0$, which is known as the Aharonov-Bohm effect.

Proposition 11. *If $\hat{\alpha}$ has constant norm $|\hat{\alpha}|$ on \widehat{M} , then the function $\mu_0: \mathbb{R} \rightarrow \mathbb{R}$ given by $\mu_0(\mathbf{B}) = \lambda_0(\mathbf{B}\hat{\alpha}, \hat{V}) - \frac{1}{2}\mathbf{B}^2|\hat{\alpha}|^2$ is concave. In particular, μ_0 is continuous, admits monotonically non-increasing left and right derivatives, and is differentiable at all but at most countably many points.*

Proof. The operator $\mathcal{H}_{\mathbf{B}\hat{\alpha}, \hat{V}} - \frac{1}{2}\mathbf{B}^2|\hat{\alpha}|^2$ with domain $C_0^\infty(\widehat{M})$ is of the form $\mathcal{K} + \mathbf{B}\mathcal{L}$, where

$$\mathcal{K}u = \frac{1}{2}\Delta u + \hat{V}u \quad \text{and} \quad \mathcal{L}u = -i\langle du, \hat{\alpha} \rangle + \frac{1}{2}d^*\hat{\alpha}u,$$

see also [AHS78, Proposition 4.11]. Since $\mathcal{K} + \mathbf{B}\mathcal{L}$ is affine linear in \mathbf{B} , its ground state μ_0 is concave in \mathbf{B} [Thi02, Theorem 3.5.21], more precisely, for $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$\begin{aligned} \mu_0(t\mathbf{B}_1 + (1-t)\mathbf{B}_2) &= \inf_{\|u\|=1} \langle (t(\mathcal{K} + \mathbf{B}_1\mathcal{L}) + (1-t)(\mathcal{K} + \mathbf{B}_2\mathcal{L}))u, u \rangle \\ &\geq \inf_{\|u\|=1} t\langle (\mathcal{K} + \mathbf{B}_1\mathcal{L})u, u \rangle + \inf_{\|u\|=1} (1-t)\langle (\mathcal{K} + \mathbf{B}_2\mathcal{L})u, u \rangle \\ &= t\mu_0(\mathbf{B}_1) + (1-t)\mu_0(\mathbf{B}_2). \end{aligned}$$

\square

In Section 5.3, we present a non-compact quotient of the Heisenberg group equipped with a left-invariant magnetic potential, such that λ_0 has countably many local minima, at which it is not differentiable. Such points are referred to as phase transitions in [HS99], which contains similar examples for magnetic Schrödinger operators on periodic graphs.

3.2. Exact case. For the remainder of this section, we consider lifts of potentials $\alpha \in \Omega^1(M, \mathbb{R})$ and $V \in C^\infty(M, \mathbb{R})$. It is well-known that $\mathcal{H}_{0,V}$ has a real-valued non-vanishing ground state. Using standard perturbation theory, Shigekawa [Shi87, Theorem 5.1 and Proposition 4.4] obtained the following generalization.

Proposition 12. ([Shi87]) *There exists $\varepsilon > 0$ such that for all $B \in (-\varepsilon, \varepsilon)$ the ground state energy $\lambda_0(B\alpha, V)$ is a simple eigenvalue of $\overline{\mathcal{H}_{B\alpha, V}}$ with non-vanishing smooth eigenfunction, and we have*

$$\lambda_0(B\alpha, V) = \inf_{\omega \in 2\pi H^1(M, \mathbb{Z})} \lambda_0 \left(0, \frac{1}{2} |B\alpha - \omega|^2 + V \right).$$

Following the proof of [Shi87, Theorem 4.3], we see that any real-valued ground state u_α of $\overline{\mathcal{H}_{\alpha, V}}$ is also a ground state of $\overline{\mathcal{H}_{0, V + \frac{1}{2}|\alpha|^2}}$ since

$$\begin{aligned} \lambda_0(\alpha, V) &= \inf_{u \in C^\infty(M, \mathbb{R}): \|u\|=1} \int_M \frac{1}{2} | -i du + u\alpha |^2 + V |u|^2 \\ &= \inf_{u \in C^\infty(M, \mathbb{R}): \|u\|=1} \int_M \frac{1}{2} |du|^2 + \left(\frac{1}{2} |\alpha|^2 + V \right) |u|^2 = \lambda_0(0, V + \frac{1}{2} |\alpha|^2). \end{aligned}$$

In particular, u_α has no zeros and the space of real-valued ground states is at most one-dimensional. In the following, we let $V = 0$ and consider the function $\lambda_0^M: \mathbb{R} \rightarrow \mathbb{R}$ given by $\lambda_0^M(B) = \lambda_0(B\alpha)$. Using Hodge decomposition of $\Omega^1(M, \mathbb{R})$, we can write $\alpha = \alpha_{cc} + df_\alpha$, where $\alpha_{cc} \in \Omega^1(M, \mathbb{R})$ is coclosed and f_α minimizes $\|\alpha - df\|$ on $C^\infty(M, \mathbb{R})$.

Lemma 13.

$$\lambda_0^M(B) \leq \frac{1}{2} B^2 \min_{f \in C^\infty(M, \mathbb{R})} \frac{\|\alpha - df\|^2}{\text{vol}(M)} = \frac{1}{2} B^2 \frac{\|\alpha_{cc}\|^2}{\text{vol}(M)}.$$

Proof. For $f \in C^\infty(M, \mathbb{R})$, we get $\lambda_0^M(B) = \lambda_0(B(\alpha - df))$ from (3.10). Thus, we can compare with the Rayleigh quotient of $\mathcal{H}_{B(\alpha-df), 0}$ at the function 1_M which is identically equal to 1 on M

$$\lambda_0^M(B) \leq \frac{\langle \mathcal{H}_{B(\alpha-df), 0} 1_M, 1_M \rangle}{\langle 1_M, 1_M \rangle} = \frac{1}{2} \frac{\|d_{B(\alpha-df)} 1_M\|^2}{\|1_M\|^2} = \frac{1}{2} B^2 \frac{\int_M |\alpha - df|^2}{\text{vol}(M)}. \quad \square$$

The following proposition is the analogue of [HS01, Proposition C] for manifolds. A proof can be obtained from the computations in [Pat01, Section 4.4].

Proposition 14. *The function λ_0^M is real analytic near $B = 0$ and satisfies*

$$\lambda_0^{M'}(0) = 0 \quad \text{and} \quad \lambda_0^{M''}(0) = \min_{f \in C^\infty(M, \mathbb{R})} \frac{\|\alpha - df\|^2}{\text{vol}(M)} = \frac{\|\alpha_{cc}\|^2}{\text{vol}(M)}.$$

Since $\overline{\mathcal{H}_{\alpha-df, V}}$ and $\overline{\mathcal{H}_{\alpha, V}}$ are gauge equivalent and therefore have the same spectrum by virtue of Theorem 10, some authors prefer to always work in the so-called Coulomb gauge given by $d^*\alpha = 0$, or equivalently $\alpha = \alpha_{cc}$, see also [RM05].

3.2.1. Twisted operators. Let $\widehat{\pi}: \widehat{M} \rightarrow M$ be a regular cover with covering transformation group G , and let $\widehat{\alpha} = \widehat{\pi}^*\alpha$ and $\widehat{V} = \widehat{\pi}^*V$ be the lifted potentials. According to Theorem 6, $\text{spec}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{V}}})$ is purely discrete or purely essential depending on whether $\text{vol}(\widehat{M}) < \infty$ or $\text{vol}(\widehat{M}) = \infty$. Let $\eta: G \rightarrow U(P)$ be a unitary representation on some separable Hilbert space P . We let E_η denote the associated flat vector bundle [Sun89], that is, E_η is the

quotient space of $\widehat{M} \times P$ by the action of G given by $\gamma(x, v) = (\gamma x, \eta(\gamma)v)$. If $[x, v]$ denotes the G -orbit of (x, v) , then the mapping $[x, v] \mapsto \widehat{\pi}(x)$ yields a vector bundle projection $E_\eta \rightarrow M$, and E_η inherits an inner product given by

$$\langle [x, v], [x, w] \rangle_\eta = \langle v, w \rangle_P.$$

Any section s of E_η can be identified with a P -valued function u_s on \widehat{M} defined via

$$s(\widehat{\pi}(x)) = [x, u_s(x)].$$

In other words, the space $\Gamma(E_\eta)$ of smooth sections of E_η can be identified with

$$(3.14) \quad \left\{ u: \widehat{M} \rightarrow P \text{ smooth} \mid u(\gamma x) = \eta(\gamma) u(x) \text{ for any } \gamma \in G, x \in \widehat{M} \right\},$$

where differentiability has to be understood with respect to the norm topology. Let $L^2(E_\eta)$ denote the completion of $\Gamma(E_\eta)$ with respect to the norm coming from the inner product

$$\langle s, t \rangle = \int_M \langle s, t \rangle_\eta,$$

and similarly for $L^2(E_\eta \otimes T^*M)$. The magnetic differential $d_\alpha: C^\infty(M) \rightarrow \Omega^1(M)$ can be extended to

$$d_{\alpha, \eta}: \Gamma(E_\eta) \rightarrow \Gamma(E_\eta \otimes T^*M).$$

We let $d_{\alpha, \eta}^*$ denote its formal adjoint and define the twisted magnetic Schrödinger operator as

$$\mathcal{H}_{\alpha, \mathbf{V}, \eta} = \frac{1}{2} d_{\alpha, \eta}^* d_{\alpha, \eta} + \mathbf{V}: \Gamma(E_\eta) \rightarrow \Gamma(E_\eta).$$

The operator $\mathcal{H}_{\alpha, \mathbf{V}, \eta}$ is sometimes called Bochner Laplacian and is known to have a unique self-adjoint extension to $L^2(E_\eta)$ [HSU80, BMS02]. Its ground state energy satisfies

$$\lambda_0(\alpha, \mathbf{V}, \eta) = \inf \text{spec}(\overline{\mathcal{H}_{\alpha, \mathbf{V}, \eta}}) = \inf_{s \in \Gamma(E_\eta) \setminus \{0\}} \frac{\int_M \frac{1}{2} \|d_{\alpha, \eta} s\|_\eta^2 + \mathbf{V} \|s\|_\eta^2}{\int_M \|s\|_\eta^2}.$$

If η is one-dimensional, then $\mathcal{H}_{\alpha, \mathbf{V}, \eta}$ can be described as follows. Lemma 7 implies that $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ commutes with the action of G . Hence, $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ maps the space (3.14) to itself, and $\mathcal{H}_{\alpha, \mathbf{V}, \eta}$ corresponds to the restriction of $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ to this space. The following lemma is proven exactly as in [Sun89].

Lemma 15. *If η is the trivial representation of G , then $(\overline{\mathcal{H}_{\alpha, \mathbf{V}, \eta}}, L^2(E_\eta))$ and $(\overline{\mathcal{H}_{\alpha, \mathbf{V}}}, L^2(M))$ are unitarily equivalent. Similarly, if η is the right regular representation on $L^2(G)$, then $(\overline{\mathcal{H}_{\alpha, \mathbf{V}, \eta}}, L^2(E_\eta))$ and $(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}}, L^2(\widehat{M}))$ are unitarily equivalent.*

3.2.2. Amenable covers. In the absence of magnetic potentials, one can use twisted Schrödinger operators to show that for any electric potential $\mathbf{V} \in C^\infty(M, \mathbb{R})$ with lift $\widehat{\mathbf{V}}$ to the regular cover \widehat{M} , we have

$$\lambda_0(0, \mathbf{V}) \leq \lambda_0(0, \widehat{\mathbf{V}}),$$

and equality holds precisely if the covering transformation group of \widehat{M} is amenable [KOS89, Proposition 1]. The special case $\mathbf{V} = 0$ and $\widehat{M} = \widehat{M}^{\text{univ}}$ is known as Brooks' theorem [Bro81], which states that $0 \in \text{spec}(\overline{\Delta^{\text{univ}}})$ if and only if $\pi_1(M)$ is amenable. We continue these developments and prove an analogue of [HS99, Theorem 2.1] for manifolds.

Theorem 16. *If \widehat{M} is a regular cover with amenable covering transformation group G , then*

$$\operatorname{spec}(\overline{\mathcal{H}_{\alpha, \mathbf{V}}}) \subseteq \operatorname{spec}(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}}).$$

Instead of working along the lines of [Sun88, Sun89], our proof follows Brooks' original approach. In essence, [Bro81, Proposition 2] says that a regular cover \widehat{M} has an amenable covering transformation group G if and only if for any $\varepsilon > 0$ and any fundamental domain F of the action of G on \widehat{M} arising from a smooth triangulation of M , there exist $\gamma_1, \gamma_2, \dots, \gamma_n \in G$ such that the compact subdomain $D = \gamma_1 F \cup \gamma_2 F \cup \dots \gamma_n F$ satisfies

$$\frac{\operatorname{vol}(\partial D)}{\operatorname{vol}(D)} < \varepsilon.$$

Following [Bro81, Section 2], one considers smooth functions that are supported inside D , and that are non-constant only in a small neighborhood of ∂D to obtain the following.

Proposition 17. *If \widehat{M} is a regular cover of the closed manifold M with amenable covering transformation group G , then for any $u \in C^\infty(M)$ and $\varepsilon > 0$ there exists $\chi_{u, \varepsilon} \in C_0^\infty(\widehat{M}, \mathbb{R})$ with*

$$(3.15) \quad \|\chi_{u, \varepsilon}\| = \sqrt{\operatorname{vol}(M)} \quad \text{and} \quad \max(\|\Delta \chi_{u, \varepsilon}\|, \|d\chi_{u, \varepsilon}\|) < \varepsilon$$

such that the lift \widehat{u} of u to \widehat{M} satisfies

$$(3.16) \quad \|\chi_{u, \varepsilon} \widehat{u}\| \geq (1 - \varepsilon) \|u\|.$$

Note that (3.15) could be weakened since

$$\|d\chi_{u, \varepsilon}\|^2 = \langle \chi_{u, \varepsilon}, \Delta \chi_{u, \varepsilon} \rangle \leq \|\chi_{u, \varepsilon}\| \|\Delta \chi_{u, \varepsilon}\|.$$

Proof of Theorem 16. It suffices to show that for any $\lambda \in \operatorname{spec}(\overline{\mathcal{H}_{\alpha, \mathbf{V}}})$, we have

$$\inf_{v \in C_0^\infty(\widehat{M}) \setminus \{0\}} \frac{\|(\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}} - \lambda)v\|}{\|v\|} = 0.$$

As M is compact and $\overline{\mathcal{H}_{\alpha, \mathbf{V}}}$ is elliptic, any $\lambda \in \operatorname{spec}(\overline{\mathcal{H}_{\alpha, \mathbf{V}}})$ is an eigenvalue of $\mathcal{H}_{\alpha, \mathbf{V}}$ with smooth eigenfunction u satisfying $\mathcal{H}_{\alpha, \mathbf{V}} u = \lambda u$. For arbitrary $\varepsilon > 0$, let $\chi_{u, \varepsilon} \in C_0^\infty(\widehat{M}, \mathbb{R})$ be as in Proposition 17, and define $v_\varepsilon = \chi_{u, \varepsilon} \widehat{u} \in C_0^\infty(\widehat{M})$, where \widehat{u} denotes the lift of u to \widehat{M} . Note that $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}} \widehat{u} = \lambda \widehat{u}$. Since $\Delta v_\varepsilon = \chi_{u, \varepsilon} \Delta \widehat{u} - 2\langle d\chi_{u, \varepsilon}, d\widehat{u} \rangle + \widehat{u} \Delta \chi_{u, \varepsilon}$, the local expression (3.3) for $\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}$ leads to

$$(\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}} - \lambda)v_\varepsilon = \frac{1}{2} \widehat{u} \Delta \chi_{u, \varepsilon} - \langle d\chi_{u, \varepsilon}, d\widehat{u} \rangle - i\langle \widehat{u} d\chi_{u, \varepsilon}, \widehat{\alpha} \rangle.$$

A similar computation appears in the proof of [Shi87, Proposition 3.2]. Since \widehat{u} and $\widehat{\alpha}$ arise from u and α , the claim follows from (3.15) and (3.16) via the estimate

$$\frac{\|(\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}} - \lambda)v_\varepsilon\|}{\|v_\varepsilon\|} \leq \frac{\varepsilon}{(1 - \varepsilon)} \frac{\frac{1}{2}\|u\|_\infty + \|du\|_\infty + \|u\|_\infty \|\alpha\|_\infty}{\|u\|}.$$

□

3.2.3. Abelian covers. We discuss Bloch-Floquet theory for lifted magnetic Schrödinger operators on abelian covers and extend results that are known for graphs [HS99] to manifolds. Our approach bases upon the study of lifted Laplacians on abelian covers in [KS00] and [Pos00, Section 3], the latter of which gives an extensive account of the underlying functional analysis.

Let $\widehat{\pi}: \widehat{M} \rightarrow M$ be a regular cover with abelian covering transformation group G . Note that $G \simeq \mathbb{Z}^{r_0} \times \mathbb{Z}_{p_1}^{r_1} \times \dots \times \mathbb{Z}_{p_k}^{r_k}$ for some $r_0, r_1, \dots, r_k \in \mathbb{N}_0$, where \mathbb{Z}_p denotes the cyclic group of order p . The irreducible unitary representations of G are one-dimensional and constitute the so-called character group $\hat{G} = \text{Hom}(G, \mathbb{S}^1) \simeq \mathbb{T}^{r_0} \times \mathbb{Z}_{p_1}^{r_1} \times \dots \times \mathbb{Z}_{p_k}^{r_k}$, which is compact with respect to its canonical topology of pointwise convergence. Recall that \widehat{M} is covered by $\widehat{M}^{\text{abel}}$, which entails a surjective homomorphism $\Phi: H_1(M, \mathbb{Z}) \rightarrow G$. Dualizing yields the following injection of compact character groups

$$(3.17) \quad \hat{\Phi}: \hat{G} \hookrightarrow \hat{H}_1(M, \mathbb{Z}) = \text{Hom}(H_1(M, \mathbb{Z}), \mathbb{S}^1).$$

The latter can be identified with the so-called Jacobian torus $H^1(M, \mathbb{R})/2\pi H^1(M, \mathbb{Z})$ via the mapping [KS88, Section 1]

$$(3.18) \quad H^1(M, \mathbb{R}) \ni [\omega] \mapsto \chi_{[\omega]} \in \hat{H}_1(M, \mathbb{Z}) \quad \text{given by} \quad \chi_{[\omega]}([\gamma]) = e^{i \int_{\gamma} \omega},$$

where $[\gamma]$ denotes the homology class of the closed curve γ . Note that if γ' is another closed curve such that $[\gamma'] = [\gamma]$, then $\gamma' \circ \gamma^{-1}$ has homotopy class in $[\pi_1(M), \pi_1(M)]$ leading to $\int_{\gamma' \circ \gamma^{-1}} \omega = 0$. Moreover, $[\omega] \in H^1(M, \mathbb{R})$ satisfies $\chi_{[\omega]} \in \hat{\Phi}(\hat{G})$ if and only if $\int_{\gamma} \omega \in 2\pi \mathbb{Z}$ for any closed curve γ in M with $[\gamma] \in \ker \Phi \subseteq H_1(M, \mathbb{Z})$, that is, if and only if $[\omega] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})$, where we defined

$$(3.19) \quad H^1(M, \widehat{M}, \mathbb{Z}) = \left\{ [\omega] \in H^1(M, \mathbb{R}) \mid \int_{\gamma} \widehat{\omega} \in \mathbb{Z} \text{ for any closed curve } \gamma \text{ in } \widehat{M} \right\}.$$

The mapping (3.18) also identifies the tangent space $T_1 \hat{G}$ at the trivial representation $\mathbf{1} \in \hat{G}$ with $\text{Hom}(G, \mathbb{R}) = \{[\omega] \in H^1(M, \mathbb{R}) \mid \widehat{\omega} \text{ is exact}\}$ as given in (2.7), see [Sun92] and [KS00, Section 2]. Essentially the same ideas work in the case of graphs [HS99, Section 3]. In the following, we use direct integral decompositions as in [RS78, Section XIII.16] and extend [KOS89, Proposition 2].

Theorem 18. *If $\widehat{\alpha}$ and $\widehat{\mathbf{V}}$ are lifts of potentials $\alpha \in \Omega^1(M, \mathbb{R})$ and $\mathbf{V} \in C^\infty(M, \mathbb{R})$ to \widehat{M} , then $(\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}}, L^2(\widehat{M}))$ allows for a direct integral decomposition, namely,*

$$(3.20) \quad L^2(\widehat{M}) = \int_{\hat{G}}^{\oplus} L^2(E_{\chi}) d\chi \quad \text{and} \quad \overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}} = \int_{\hat{G}}^{\oplus} \overline{\mathcal{H}_{\alpha + \omega_{\chi}, \mathbf{V}}} d\chi,$$

where E_{χ} denotes the flat vector bundle associated with $\chi \in \hat{G}$, and $d\chi$ denotes the normalized Haar measure on \hat{G} . Moreover, if $\chi \in \hat{G}$ is represented by $\omega_{\chi} \in \Omega^1(M, \mathbb{R})$ via (3.17) and (3.18), then $(\overline{\mathcal{H}_{\alpha, \mathbf{V}, \chi}}, L^2(E_{\chi}))$ is unitarily equivalent to $(\overline{\mathcal{H}_{\alpha + \omega_{\chi}, \mathbf{V}}}, L^2(M))$.

Proof. For each $\chi \in \hat{G}$, we use (3.14) to identify $\Gamma(E_{\chi})$ with the space of smooth χ -periodic functions on \widehat{M} , on which $\overline{\mathcal{H}_{\alpha, \mathbf{V}, \chi}}$ acts as $\overline{\mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{V}}}}$. With respect to this identification, we have $\|u\|^2 = \int_F |u|^2 d\text{vol}$ for $u \in \Gamma(E_{\chi}) \subset C^\infty(\widehat{M})$, where F is a fundamental domain for the action

of G on \widehat{M} . Following [Don81, Theorem 3.3], [KOS89, Proposition 2] and [Pos00, Section 3.3], we verify that the Gelfand transform

$$\mathcal{I}: C_0^\infty(\widehat{M}) \rightarrow \int_{\widehat{G}}^\oplus L^2(E_\chi) d\chi \quad \text{given by} \quad (\mathcal{I}u)_\chi(x) = \sum_{\gamma \in G} \overline{\chi(\gamma)} u(\gamma x)$$

extends to the desired isometry. For $\gamma \in G$, we have

$$(\mathcal{I}u)_\chi(\gamma x) = \sum_{\gamma' \in G} \overline{\chi(\gamma')} u(\gamma' \gamma x) = \sum_{\gamma'' \in G} \overline{\chi(\gamma'')} \overline{\chi(\gamma^{-1})} u(\gamma'' x) = \chi(\gamma) (\mathcal{I}u)_\chi(x).$$

In other words, $(\mathcal{I}u)_\chi \in \Gamma(E_\chi)$. Using the orthogonality of characters, we obtain

$$\begin{aligned} \|\mathcal{I}u\|^2 &= \int_{\widehat{G}} \|(\mathcal{I}u)_\chi\|^2 d\chi = \int_{\widehat{G}} \int_F \left(\sum_{\gamma \in G} \overline{\chi(\gamma)} \gamma^* u \right) \left(\sum_{\gamma' \in G} \chi(\gamma') \overline{\gamma'^* u} \right) d\text{vol} d\chi \\ &= \int_F \sum_{\gamma, \gamma' \in G} \gamma^* u \overline{\gamma'^* u} \int_{\widehat{G}} \overline{\chi(\gamma)} \chi(\gamma') d\chi d\text{vol} \\ &= \int_F \sum_{\gamma \in G} |\gamma^* u|^2 d\text{vol} = \int_{\widehat{M}} |u|^2 d\text{vol} = \|u\|_{L^2(\widehat{M})}^2. \end{aligned}$$

The inverse of \mathcal{I} reads

$$(\mathcal{I}^{-1}(u_\chi)_{\chi \in \widehat{G}})(x) = \int_{\widehat{G}} u_\chi(x) d\chi,$$

and we have $\mathcal{I} \mathcal{H}_{\widehat{\alpha}, \widehat{\mathbf{v}}} \mathcal{I}^{-1}(u_\chi)_{\chi \in \widehat{G}} = (\mathcal{H}_{\alpha, \mathbf{v}, \chi} u_\chi)_{\chi \in \widehat{G}}$ as claimed. In the following, let $\omega_\chi \in \Omega^1(M, \mathbb{R})$ represent $\chi \in \widehat{G}$, in particular, its lift $\widehat{\omega}_\chi$ satisfies $\int_\gamma \widehat{\omega}_\chi \in 2\pi \mathbb{Z}$ for any closed curve γ in \widehat{M} . Following [Sun85, Proposition 4], we choose some fixed reference point $x_0 \in \widehat{M}$ and define $\widehat{u}_\chi: \widehat{M} \rightarrow \mathbb{S}^1$ as

$$\widehat{u}_\chi(x) = e^{i \int_{x_0}^x \widehat{\omega}_\chi}.$$

Note that \widehat{u}_χ is well-defined, satisfies $d\widehat{u}_\chi = i \widehat{u}_\chi \widehat{\omega}_\chi$, and is χ -periodic since for $\gamma \in G$, we have

$$\widehat{u}_\chi(\gamma x) = e^{i \int_x^{\gamma x} \widehat{\omega}_\chi} e^{i \int_{x_0}^x \widehat{\omega}_\chi} = \chi(\gamma) \widehat{u}_\chi(x).$$

Using that $\widehat{u}_\chi \in \Gamma(E_\chi)$, we define unitary maps $\mathcal{U}_\chi: C^\infty(M) \rightarrow \Gamma(E_\chi)$ and $\mathcal{W}_\chi: \Omega^1(M) \rightarrow \Gamma(E_\chi \otimes T^*M)$ as $\mathcal{U}_\chi u = \widehat{u}_\chi \widehat{u}$ and $\mathcal{W}_\chi \omega = \widehat{u}_\chi \widehat{\omega}$, where \widehat{u} and $\widehat{\omega}$ denote the lifts of u and ω to \widehat{M} , respectively. For $u \in C^\infty(M)$, we have

$$\begin{aligned} d_{\alpha, \chi} \mathcal{U}_\chi u = d_{\widehat{\alpha}}(\widehat{u}_\chi \widehat{u}) &= -i \widehat{u}_\chi d\widehat{u} - i \widehat{u} d\widehat{u}_\chi + \widehat{u}_\chi \widehat{u} \widehat{\alpha} \\ &= -i \widehat{u}_\chi d\widehat{u} + \widehat{u}_\chi \widehat{u} (\widehat{\omega}_\chi + \widehat{\alpha}) = \widehat{u}_\chi d_{\widehat{\alpha} + \widehat{\omega}_\chi} \widehat{u} = \mathcal{W}_\chi d_{\alpha + \omega_\chi} u. \end{aligned}$$

We take formal adjoints to obtain

$$\mathcal{U}_\chi^{-1} d_{\alpha, \chi}^* = \mathcal{U}_\chi^* d_{\alpha, \chi}^* = d_{\alpha + \omega_\chi}^* \mathcal{W}_\chi^* = d_{\alpha + \omega_\chi}^* \mathcal{W}_\chi^{-1}.$$

Since $\mathcal{U}_\chi^{-1} \widehat{\mathbf{V}} \mathcal{U}_\chi = \mathbf{V}$ trivially holds on $C^\infty(M)$, the analogue of [HS99, Lemma 3.1] for manifolds follows, namely,

$$\mathcal{U}_\chi^{-1} \overline{\mathcal{H}_{\alpha, \mathbf{V}, \chi}} \mathcal{U}_\chi = \overline{\mathcal{H}_{\alpha + \omega_\chi, \mathbf{V}}}.$$

□

Theorem 19. *The spectrum of $\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}$ as in Theorem 18 has band structure*

$$(3.21) \quad \text{spec}(\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}) = \bigcup_{[\omega] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})} \text{spec}(\overline{\mathcal{H}_{\alpha-\omega, V}}) = \bigcup_{k \in I} [a_k, b_k],$$

where $H^1(M, \widehat{M}, \mathbb{Z})$ is given in (3.19), and we either have $I = \{1, 2, \dots, N\}$ and $b_k < a_{k+1}$ for $k \in I \setminus \{N\}$ as well as $b_N = \infty$, or $I = \mathbb{N}$ and $b_k < a_{k+1}$ for all $k \in I$ as well as $a_k \nearrow \infty$. Moreover,

$$\lambda_0(\hat{\alpha}, \hat{V}) = \min_{[\omega] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})} \lambda_0(\alpha - \omega, V).$$

In particular, $\hat{\alpha}$ has no diamagnetic effect, that is, $\lambda_0(\hat{\alpha}, \hat{V}) = \lambda_0(0, \hat{V})$, if and only if α is closed and $[\alpha] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})$.

Proof. For arbitrary $\omega \in \Omega^1(M, \mathbb{R})$, the spectrum of $\overline{\mathcal{H}_{\alpha+\omega, V}}$ is discrete with smooth eigenfunctions [Shi87, Theorem 2.1]. Thus, the spectra of the twisted operators $\overline{\mathcal{H}_{\alpha, V, \chi}}$ in the direct integral decomposition (3.20) are given as unbounded sequences

$$\lambda_0(\chi) \leq \lambda_1(\chi) \leq \lambda_2(\chi) \leq \dots$$

Perturbation theory as in [KS60, Theorem 2] shows that these eigenvalue functions $\lambda_k: \hat{G} \rightarrow \mathbb{R}$ are continuous, see also [KS88, KOS89, KS00]. The claim (3.21) now follows from the theory of direct integrals and compactness of \hat{G} , namely,

$$\text{spec}(\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}) = \bigcup_{k \in \mathbb{N}_0} \bigcup_{\chi \in \hat{G}} \lambda_k(\chi) = \bigcup_{k \in \mathbb{N}_0} \lambda_k(\hat{G}).$$

The remaining statements are consequences of Theorem 10 and the fact that each $\chi \in \hat{G}$ is represented by some $[\omega] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})$. \square

As an example, take the n -fold covering of $M = \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ by $\widehat{M} = \mathbb{S}^1$ with projection $\widehat{\pi}(z) = z^n$. If $n > 1$ and $\varphi \in (0, 2\pi)$ denotes the angular coordinate of $z = e^{i\varphi} \in M \setminus \{1\}$, then $\frac{1}{n} d\varphi$ extends to a closed 1-form α on M satisfying $[\alpha] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z}) \setminus 2\pi H^1(\widehat{M}, \mathbb{Z})$. For arbitrary $V \in C^\infty(M, \mathbb{R})$, Theorem 10 and Theorem 19 thus yield

$$\lambda_0(\hat{\alpha}, \hat{V}) = \lambda_0(0, V) < \lambda_0(\alpha, V),$$

which contrasts the non-magnetic case in which $\lambda_0(0, \hat{V}) \geq \lambda_0(0, V)$ for any cover \widehat{M} [KOS89, Corollary of Proposition 1]. We present another example of this type in Section 5.3. The following corollary of Theorem 16 and Theorem 19 should be compared with Theorem 2.

Corollary 20. *If \widetilde{M} is a regular amenable cover of M with abelian subcover \widehat{M} , and if $(\tilde{\alpha}, \tilde{V})$ and $(\hat{\alpha}, \hat{V})$ denote the lifts of $\alpha \in \Omega^1(M, \mathbb{R})$ and $V \in C^\infty(M, \mathbb{R})$ to \widetilde{M} and \widehat{M} , respectively, then*

$$\text{spec}(\overline{\mathcal{H}_{\alpha, V}}) \subseteq \text{spec}(\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}) = \bigcup_{[\omega] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})} \text{spec}(\overline{\mathcal{H}_{\alpha-\omega, V}}) \subseteq \text{spec}(\overline{\mathcal{H}_{\tilde{\alpha}, \tilde{V}}}).$$

In particular, if $\pi_1(M)$ is amenable, then

$$\lambda_0(\alpha, V) \geq \lambda_0(\hat{\alpha}^{\text{abel}}, \hat{V}^{\text{abel}}) = \min_{[\omega] \in H^1(M, \mathbb{R})} \lambda_0(\alpha - \omega, V) \geq \lambda_0(\hat{\alpha}^{\text{univ}}, \hat{V}^{\text{univ}}).$$

Henceforth, let $\mathbf{V} = 0$. Using Hodge theory, we identify $H^1(M, \mathbb{R})$ with the space of harmonic 1-forms on M . In particular, let $\omega_1, \omega_2, \dots, \omega_{r_0}$ be harmonic 1-forms that represent an orthonormal basis of $T_1 \hat{G} \simeq \text{Hom}(G, \mathbb{R}) \subseteq H^1(M, \mathbb{R})$. Let $\alpha \in \Omega^1(M, \mathbb{R})$ have Hodge decomposition

$$\alpha = \alpha_h^\wedge + \alpha_h^\perp + d^* \beta_\alpha + df_\alpha,$$

where $\beta_\alpha \in \Omega^2(M, \mathbb{R})$, $f_\alpha \in C^\infty(M, \mathbb{R})$, and α_h^\wedge as well as α_h^\perp are harmonic such that $\alpha_h^\wedge \in T_1 \hat{G}$ and $\alpha_h^\perp \in T_1 \hat{G}^\perp$. If $\alpha_h^\perp + d^* \beta_\alpha = 0$, then $[\mathbf{B} \alpha] = [\mathbf{B} \alpha_h^\wedge] \in 2\pi H^1(M, \widehat{M}, \mathbb{Z})$ for any $\mathbf{B} \in \mathbb{R}$, in particular, $\lambda_0(\mathbf{B} \hat{\alpha}) = \lambda_0(0)$. Thus, we assume that $\alpha_h^\perp + d^* \beta_\alpha \neq 0$ and consider the finite-dimensional vector space

$$X_\alpha = \mathbb{R}(\alpha_h^\perp + d^* \beta_\alpha) \oplus \bigoplus_{j=1}^{r_0} \mathbb{R} \omega_j \subseteq \Omega^1(M, \mathbb{R}).$$

The following generalization of Proposition 14 to abelian covers is an extended version of [HS99, Theorem 1.2] for manifolds.

Theorem 21. *Let $\lambda_{0, X_\alpha}: X_\alpha \rightarrow \mathbb{R}$ be given by $\lambda_{0, X_\alpha}(\omega) = \lambda_0(\omega)$. In a neighborhood of $0 \in X_\alpha$, the function λ_{0, X_α} is real analytic and has positive definite Hessian. In particular, $\lambda_0^{\widehat{M}}: \mathbb{R} \rightarrow \mathbb{R}$ given by $\lambda_0^{\widehat{M}}(\mathbf{B}) = \lambda_0(\mathbf{B} \hat{\alpha})$ is real analytic near $\mathbf{B} = 0$ and satisfies*

$$\lambda_0^{\widehat{M}''}(0) = \min_{[\omega] \in \text{Hom}(G, \mathbb{R})} \frac{\|\alpha - \omega\|^2}{\text{vol}(M)} = \frac{\|\alpha_h^\perp\|^2 + \|d^* \beta_\alpha\|^2}{\text{vol}(M)},$$

where $\text{Hom}(G, \mathbb{R})$ denotes the vector space (2.7).

Proof. The claimed real analyticity of λ_{0, X_α} follows from perturbation theory as $\lambda_{0, X_\alpha}(0) = 0$ is a simple eigenvalue of the Laplacian $2\mathcal{H}_{0,0}$. Moreover, Proposition 14 says that

$$\text{Hess } \lambda_{0, X_\alpha}(0)[\omega, \omega] = \frac{\|\omega\|^2}{\text{vol}(M)},$$

which implies positive definiteness near $0 \in X_\alpha$. The statements concerning $\lambda_0^{\widehat{M}}$ follow from [HS99, Lemma 4.2] as in the proof of [HS99, Theorem 4.3]. More precisely, we can assume that $f_\alpha = 0$ using (3.10). The proof of Theorem 19 revealed that

$$\lambda_0^{\widehat{M}}(\mathbf{B}) = \min_{\chi \in \hat{G}} \lambda_0(\mathbf{B} \alpha + \omega_\chi),$$

where the closed form ω_χ gives rise to $\chi \in \hat{G}$ via (3.17) and (3.18). We verify that for each open neighborhood U of $\mathbf{1} \in \hat{G}$, there exists $\varepsilon_U > 0$ such that $|\mathbf{B}| < \varepsilon_U$ implies

$$\lambda_0^{\widehat{M}}(\mathbf{B}) = \min_{\chi \in U} \lambda_0(\mathbf{B} \alpha + \omega_\chi).$$

Otherwise, we can find sequences $(\mathbf{B}_j)_{j \in \mathbb{N}} \searrow 0$ and $(\chi_j)_{j \in \mathbb{N}} \subset \hat{G} \setminus U$ with

$$\lambda_0^{\widehat{M}}(\mathbf{B}_j) = \lambda_0(\mathbf{B}_j \alpha + \omega_{\chi_j}) \leq \lambda_0(\mathbf{B}_j \alpha).$$

Note that $\lambda_0(\mathbf{B}_j \alpha) \rightarrow 0$ by virtue of Proposition 14. Since $\hat{G} \setminus U$ is compact, a subsequence of $(\chi_j)_{j \in \mathbb{N}}$ converges to some $\chi_0 \in \hat{G} \setminus U$ with $\lambda_0(\omega_{\chi_0}) = 0$. However, Theorem 10 thus implies that $[\omega_{\chi_0}] \in 2\pi H^1(M, \mathbb{Z})$ which contradicts $\chi_0 \neq \mathbf{1}$. Since $\hat{G} \simeq \mathbb{T}^{r_0} \times \mathbb{Z}_{p_1}^{r_1} \times \dots \times \mathbb{Z}_{p_k}^{r_k}$, we

may regard any open neighborhood W of $0 \in \bigoplus_{j=1}^{r_0} \mathbb{R} \omega_j \simeq T_1 \hat{G}$ as an open neighborhood of $\mathbf{1} \in \hat{G}$, and find $\varepsilon_W > 0$ such that $|\mathbf{B}| < \varepsilon_W$ implies

$$\widehat{\lambda}_0^{\widehat{M}}(\mathbf{B}) = \min_{\omega \in W} \lambda_{0, X_{\alpha}}(\mathbf{B} \alpha + \omega).$$

We imitate the proof of [HS99, Theorem 4.3] and apply [HS99, Lemma 4.2] with $e_0 = \alpha$, $e_j = \omega_j$ for $j = 1, 2, \dots, r_0$, and $f = \lambda_{0, X_{\alpha}}$ for sufficiently small $|\mathbf{B}|$ to obtain real analyticity of $\widehat{\lambda}_0^{\widehat{M}}$ near $\mathbf{B} = 0$ as well as

$$\begin{aligned} \widehat{\lambda}_0^{\widehat{M}''}(0) &= \frac{1}{\text{vol}(M)} \text{Det} \begin{pmatrix} \langle \alpha, \alpha \rangle & \langle \alpha, \omega_1 \rangle & \langle \alpha, \omega_2 \rangle & \dots & \langle \alpha, \omega_{r_0} \rangle \\ \langle \alpha, \omega_1 \rangle & 1 & 0 & & 0 \\ \langle \alpha, \omega_2 \rangle & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ \langle \alpha, \omega_n \rangle & 0 & 0 & \dots & 1 \end{pmatrix} \\ &= \frac{1}{\text{vol}(M)} \left(\|\alpha\|^2 - \sum_{j=1}^{r_0} \langle \alpha, \omega_j \rangle^2 \right) = \min_{[\omega] \in \text{Hom}(G, \mathbb{R})} \frac{\|\alpha - \omega\|^2}{\text{vol}(M)}. \end{aligned}$$

□

4. MAÑÉ'S CRITICAL VALUE AND THE GROUND STATE ENERGY

Let M be a connected closed manifold with potentials $\alpha \in \Omega^1(M, \mathbb{R})$ and $V \in C^\infty(M, \mathbb{R})$. Recall that Mañé's critical value of the corresponding lifted Lagrangian $L_{\hat{\alpha}, \hat{V}}$ on a regular cover \widehat{M} of M is given by [CIPP98, Theorem A]

$$(4.1) \quad c(L_{\hat{\alpha}, \hat{V}}) = \inf_{f \in C^\infty(\widehat{M}, \mathbb{R})} \sup_{x \in \widehat{M}} \frac{1}{2} |\hat{\alpha} + df|_x^2 + \hat{V}(x).$$

For the trivial cover $\widehat{M} = M$, the following is known.

Theorem 22. ([Pat01, Theorem B]) *We have $\lambda_0(\alpha, V) \leq c(L_{\alpha, V})$.*

The preceding theorem is proven by evaluating the Rayleigh quotient of $\mathcal{H}_{\alpha, V}$ at a gauged version of the constant function 1_M given by $1_M(x) = 1$ for $x \in M$. For arbitrary $\varepsilon > 0$, let $f_\varepsilon \in C^\infty(M, \mathbb{R})$ be such that

$$\sup_{x \in M} \frac{1}{2} |\alpha + df_\varepsilon|_x^2 + V(x) < c(L_{\alpha, V}) + \varepsilon.$$

Considering α in the gauge $\alpha + df_\varepsilon$, we obtain from (3.10)

$$\lambda_0(\alpha, V) = \lambda_0(\alpha + df_\varepsilon, V) \leq \frac{\langle \mathcal{H}_{\alpha + df_\varepsilon, V} 1_M, 1_M \rangle}{\langle 1_M, 1_M \rangle} = \frac{\int_M \frac{1}{2} |\alpha + df_\varepsilon|^2 + V}{\text{vol}(M)} < c(L_{\alpha, V}) + \varepsilon.$$

Note that the proof relies heavily on $1_M \in L^2(M)$. Brooks [Bro81] was the first to discover that for amenable covers \widehat{M} , there exist replacements $\chi_\varepsilon \in C_0^\infty(\widehat{M}, \mathbb{R})$ for $1_{\widehat{M}}$ as described by Proposition 17. In combination with Paternain's approach, we obtain the following generalization of Theorem 22 to amenable covers, including the monopole case.

Theorem 23. *Let \widehat{M} be a regular cover of M with amenable covering transformation group. If $\hat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$ and $\hat{V} \in C^\infty(\widehat{M}, \mathbb{R})$ are potentials with $\inf \hat{V} > -\infty$, then the associated Lagrangian $L_{\hat{\alpha}, \hat{V}}$ and the associated magnetic Schrödinger operator $\mathcal{H}_{\hat{\alpha}, \hat{V}}$ satisfy*

$$(4.2) \quad \lambda_0(\overline{\mathcal{H}_{\hat{\alpha}, \hat{V}}}) = \lambda_0(\hat{\alpha}, \hat{V}) \leq c(L_{\hat{\alpha}, \hat{V}}).$$

A proof for the exact case in which $\hat{\alpha}$ and \hat{V} are lifts of potentials α and V on M reads

$$\lambda_0(\hat{\alpha}, \hat{V}) = \min_{\substack{\omega \in \Omega^1(M, \mathbb{R}): \\ \hat{\omega} \text{ is exact}}} \lambda_0(\hat{\alpha} - \hat{\omega}, \hat{V}) \leq \min_{\substack{\omega \in \Omega^1(M, \mathbb{R}): \\ \hat{\omega} \text{ is exact}}} \lambda_0(\alpha - \omega, V) \leq \min_{\substack{\omega \in \Omega^1(M, \mathbb{R}): \\ \hat{\omega} \text{ is exact}}} c(L_{\alpha - \omega, V}) = c(L_{\hat{\alpha}, \hat{V}}),$$

where we used Theorem 10, Theorem 16, Theorem 22 and Theorem 2, respectively.

Proof of Theorem 23. Since (4.2) trivially holds for $c(L_{\hat{\alpha}, \hat{V}}) = \infty$, we assume $c(L_{\hat{\alpha}, \hat{V}}) < \infty$. For arbitrary $\varepsilon > 0$, we can find $f_\varepsilon \in C^\infty(\widehat{M}, \mathbb{R})$ such that

$$\sup_{x \in \widehat{M}} \frac{1}{2} |\hat{\alpha} + df_\varepsilon|_x^2 + \hat{V}(x) < c(L_{\hat{\alpha}, \hat{V}}) + \varepsilon.$$

Let $\chi_{f_\varepsilon, \varepsilon} \in C_0^\infty(\widehat{M}, \mathbb{R})$ be as in Proposition 17. As before, we use (3.10) to deduce

$$\begin{aligned}
\lambda_0(\widehat{\alpha}, \widehat{V}) &= \lambda_0(\widehat{\alpha} + df_\varepsilon, \widehat{V}) \leq \frac{\langle \mathcal{H}_{\widehat{\alpha} + df_\varepsilon, \widehat{V}} \chi_{f_\varepsilon, \varepsilon}, \chi_{f_\varepsilon, \varepsilon} \rangle}{\langle \chi_{f_\varepsilon, \varepsilon}, \chi_{f_\varepsilon, \varepsilon} \rangle} \\
&= \frac{1}{\text{vol}(M)} \left(\int_M \frac{1}{2} |-id \chi_{f_\varepsilon, \varepsilon} + \chi_{f_\varepsilon, \varepsilon}(\widehat{\alpha} + df_\varepsilon)|^2 + \chi_{f_\varepsilon, \varepsilon}^2 \widehat{V} \right) \\
&= \frac{1}{\text{vol}(M)} \left(\frac{1}{2} \|d \chi_{f_\varepsilon, \varepsilon}\|^2 + \int_M \chi_{f_\varepsilon, \varepsilon}^2 \left(\frac{1}{2} |\widehat{\alpha} + df_\varepsilon|^2 + \widehat{V} \right) \right) \\
&< \frac{1}{2} \frac{\varepsilon^2}{\text{vol}(M)} + \mathbf{c}(L_{\widehat{\alpha}, \widehat{V}}) + \varepsilon. \quad \square
\end{aligned}$$

In order to illustrate the relationship between λ_0 and \mathbf{c} , let \widehat{M} be a regular cover of the closed manifold M with magnetic potential $\widehat{\alpha} \in \Omega^1(\widehat{M}, \mathbb{R})$ having constant norm $|\widehat{\alpha}|$. In particular, $\lambda_0^{\widehat{M}}: \mathbb{R} \rightarrow \mathbb{R}$ given by $\lambda_0^{\widehat{M}}(\mathbf{B}) = \lambda_0(\mathbf{B}\widehat{\alpha})$ is continuous by virtue of Proposition 11. For the Lagrangian $L_{\mathbf{B}\widehat{\alpha}} = L_{\mathbf{B}\widehat{\alpha}, 0}$, one easily obtains $\mathbf{c}(L_{\mathbf{B}\widehat{\alpha}}) = \mathbf{B}^2 \mathbf{c}(L_{\widehat{\alpha}})$ from (4.1). If \widehat{M} is a non-amenable cover, then the extended theorem of Brooks in [KOS89] states that $\lambda_0^{\widehat{M}}(0) > 0$, which implies $\lambda_0^{\widehat{M}}(\mathbf{B}) = \lambda_0(\mathbf{B}\widehat{\alpha}) > \mathbf{c}(L_{\mathbf{B}\widehat{\alpha}})$ near $\mathbf{B} = 0$. In Section 5.2, we give an explicit example of this type, where $\widehat{\alpha}$ is the lift of some $\alpha \in \Omega^1(M, \mathbb{R})$. In Section 5.3, we present a similar example with nilpotent and therefore amenable $\pi_1(M)$, such that $\lambda_0^{\widehat{M}^{\text{univ}}}(\mathbf{B}) \leq \lambda_0^{\widehat{M}^{\text{abel}}}(\mathbf{B})$ with equality only if $|\mathbf{B}| \leq \frac{1}{2}$ or $|\mathbf{B}| - \frac{1}{4} \in 2\pi\mathbb{Z}$ contrasting Theorem 2 which yields $\mathbf{c}(L_{\mathbf{B}\widehat{\alpha}}^{\text{univ}}) = \mathbf{c}(L_{\mathbf{B}\widehat{\alpha}}^{\text{abel}})$ for all $\mathbf{B} \in \mathbb{R}$, see also Corollary 20. It is worth mentioning that all examples of compact homogeneous spaces $M = \Lambda \backslash \Gamma$ in Section 5, where Γ is a Lie group with cocompact lattice $\Lambda \subset \Gamma$ and left-invariant potential descending to $\alpha \in \Omega^1(M, \mathbb{R})$, satisfy

$$(4.3) \quad \lambda_0^{\widehat{M}}(\mathbf{B}) = \lambda_0^{\widehat{M}}(0) + \mathbf{B}^2 \mathbf{c}(L_{\widehat{\alpha}}) \quad \text{near } \mathbf{B} = 0$$

on the universal cover and various intermediate covers, irrespective of amenability of $\pi_1(M)$. In particular, $\lambda_0^{\widehat{M}}$ is a quadratic function near $\mathbf{B} = 0$ with $\lambda_0^{\widehat{M}''}(0) = 2 \mathbf{c}(L_{\widehat{\alpha}})$. Note that each such α coincides with its coclosed part α_{cc} since the Hodge dual $*\alpha$ as well as $d*\alpha$ lift to left-invariant forms on Γ for which reason $d*\alpha$ is a constant multiple of the non-exact volume form on M , hence, $d*\alpha = -*d*\alpha = 0$. For the trivial cover $\widehat{M} = M$ and arbitrary $\alpha \in \Omega^1(M, \mathbb{R})$, Paternain [Pat01, Proposition 4.2] showed that (4.3) holds precisely if $\mathbf{c}(L_\alpha)$ coincides with

$$h(L_\alpha) = \inf_{f \in C^\infty(M, \mathbb{R})} \frac{\int_M \frac{1}{2} |\alpha + df|^2}{\text{vol}(M)} = \frac{1}{2} \frac{\|\alpha_{\text{cc}}\|^2}{\text{vol}(M)}.$$

This condition is satisfied if M is a homogeneous space with $\alpha \in \Omega^1(M, \mathbb{R})$ as above or, more generally, if $|\alpha_{\text{cc}}|_x^2 = 2 h(L_\alpha)$ for all $x \in M$ as this implies

$$h(L_\alpha) \leq \mathbf{c}(L_\alpha) \leq \sup_{x \in M} \frac{1}{2} |\alpha_{\text{cc}}|_x^2 = h(L_\alpha).$$

In the former case, Theorem 2 and Theorem 21 give $\lambda_0^{\widehat{M}''}(0) = 2 \mathbf{c}(L_{\widehat{\alpha}})$ on covers with abelian covering transformation group G for which $\text{Hom}(G, \mathbb{R})$ is generated by left-invariant forms.

5. HOMOGENEOUS EXAMPLES AND THE 3-BODY PROBLEM

In this section, we study magnetic Schrödinger operators on covers \widehat{M} of compact homogeneous spaces $M = \Lambda \backslash \Gamma$, and compare with the corresponding classical data. In each case, Γ is a Lie group equipped with a left-invariant metric and a left-invariant magnetic field $\beta \in \Omega^2(\Gamma, \mathbb{R})$, and $\Lambda \subset \Gamma$ is a cocompact lattice, that is, a discrete subgroup such that the quotient $\Lambda \backslash \Gamma$ is compact. According to Theorem 5, the corresponding magnetic Schrödinger operators are essentially self-adjoint. Therefore, we use the same symbol $\mathcal{H}^{\widehat{M}}$ for the respective operator and its self-adjoint closure with the exception of the last subsection, where we discuss the quantum analogue of the classical 3-body problem.

5.1. Tori. We consider $\mathbb{T}^n = \mathbb{Z}^n \backslash \mathbb{R}^n = \mathbb{R}^n / \mathbb{Z}^n$ with its usual flat metric coming from the Euclidean metric on \mathbb{R}^n . Note that the lift of any left-invariant 2-form β on \mathbb{T}^n to the universal cover \mathbb{R}^n takes the form $\beta^{\mathbb{R}^n} = \sum_{j < k} B_{jk} dy^j \wedge dy^k$ for some $B_{jk} \in \mathbb{R}$. We briefly recall that there exists an isometry $Q \in O(n)$ of \mathbb{R}^n with respect to which

$$(5.1) \quad \beta^{\mathbb{R}^n} = \sum_{j=1}^{\frac{r(B)}{2}} \lambda_j dx^{2j-1} \wedge dx^{2j} \text{ with all } \lambda_j \neq 0,$$

where (x_1, x_2, \dots, x_n) denote the linear coordinates given by $Q: \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $r(B) \in 2\mathbb{N}_0$ denotes the rank of the skew-symmetric matrix $B = (B_{jk})_{j,k=1}^n$ with entries $B_{jk} = -B_{kj}$ for $j > k$. Since iB is hermitian, it has real eigenvalues and can be diagonalized by a unitary matrix each of whose columns u satisfies $(iB)u = \lambda u$ for some $\lambda \in \mathbb{R}$. Complex conjugation leads to $(iB)\bar{u} = -\lambda\bar{u}$. Hence, iB can be diagonalized by a unitary matrix having columns

$$(u_1, \bar{u}_1, u_2, \bar{u}_2, \dots, u_{\frac{r(B)}{2}}, \bar{u}_{\frac{r(B)}{2}}, u_{r(B)+1}, u_{r(B)+2}, \dots, u_n),$$

where $u_j \in \ker B \cap \mathbb{R}^n$ for $j > r(B)$. Moreover, B has eigenvalues of the form

$$(-i\lambda_1, i\lambda_1, -i\lambda_2, i\lambda_2, \dots, -i\lambda_{\frac{r(B)}{2}}, i\lambda_{\frac{r(B)}{2}}, 0, 0, \dots, 0),$$

where all $\lambda_j \neq 0$. If $u_j = v_j + iw_j$ with real and imaginary parts $v_j, w_j \in \mathbb{R}^n$, then we can use the orthonormal basis $\{v_j \pm w_j\}_{j \leq \frac{r(B)}{2}} \cup \{u_j\}_{j > r(B)}$ of \mathbb{R}^n to obtain (5.1). Since $H^1(\mathbb{R}^n, \mathbb{R})$ is trivial, any magnetic potential of (5.1) is gauge equivalent to

$$(5.2) \quad \alpha = \sum_{j=1}^{\frac{r(B)}{2}} \lambda_j x_{2j-1} dx^{2j}.$$

If $\beta^{\mathbb{R}^n}$ has bounded primitives, then, by amenability of $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$ and [Pat06, Lemma 5.3], it also has left-invariant ones, that is, $[\beta] = 0 \in H^2(\mathbb{T}^n, \mathbb{R})$, and we have $\beta = 0$. The critical value on \mathbb{R}^n is therefore given by

$$c(H_\alpha^{\mathbb{R}^n}) = \begin{cases} 0 & \text{if } \beta = 0 \\ \infty & \text{otherwise,} \end{cases}$$

where $H_\alpha^{\mathbb{R}^n}: T^*\mathbb{R}^n \rightarrow \mathbb{R}$ is the magnetic Hamiltonian corresponding to the potential (5.2). If β is symplectic, meaning $r(B) = n$, then a similarly drastic behavior can be observed for the spectrum of the corresponding magnetic Schrödinger operator. Following [FH10, Section 1.4.3], we let $\text{Tr}^+(\beta) = \sum_{j=1}^{\frac{r(B)}{2}} |\lambda_j|$.

Theorem 24. *The closure of*

$$\mathcal{H}_\alpha^{\mathbb{R}^n} = \sum_{j=1}^{\frac{r(B)}{2}} \left(\left(\frac{1}{i} \frac{\partial}{\partial x_{2j-1}} \right)^2 + \left(\frac{1}{i} \frac{\partial}{\partial x_{2j}} + \lambda_j x_{2j-1} \right)^2 \right) + \sum_{j=r(B)+1}^n \left(\frac{1}{i} \frac{\partial}{\partial x_j} \right)^2$$

with initial domain $C_0^\infty(\mathbb{R}^n)$ has the purely essential spectrum

$$\text{spec}(\mathcal{H}_\alpha^{\mathbb{R}^n}) = \text{spec}_{\text{ess}}(\mathcal{H}_\alpha^{\mathbb{R}^n}) = \begin{cases} [\text{Tr}^+(\beta), \infty) & \text{if } r(B) < n \\ \left\{ \sum_{j=1}^{\frac{r(B)}{2}} |\lambda_j| (2k_j + 1) \mid k_j \in \{0, 1, 2, \dots\} \right\} & \text{if } \beta \text{ is symplectic.} \end{cases}$$

In any case, we have $\lambda_0(\mathcal{H}_\alpha^{\mathbb{R}^n}) = \text{Tr}^+(\beta)$.

Proof. The theorem is a special case of [Shi91, Theorem 2.2 and Theorem 2.4], which deal with asymptotically constant magnetic fields on \mathbb{R}^n . For the reader's convenience, we provide a less involved proof along the lines of [FH10, Section 1.4.3]. Due to Theorem 6, we know that $\mathcal{H}_\alpha^{\mathbb{R}^n}$ has purely essential spectrum. Since $\mathcal{H}_\alpha^{\mathbb{R}^n}$ is a sum of operators as in [RS80, Theorem VIII.33], we see that

$$\left(\bigotimes_{j=1}^{\frac{r(B)}{2}} C_0^\infty(\mathbb{R}^2) \right) \otimes C_0^\infty(\mathbb{R}^{n-r(B)}) \subset \left(\bigotimes_{j=1}^{\frac{r(B)}{2}} L^2(\mathbb{R}^2) \right) \otimes L^2(\mathbb{R}^{n-r(B)}) \simeq L^2(\mathbb{R}^n)$$

is a core, and it suffices to study the self-adjoint model operators $\mathcal{H}_\lambda^{\mathbb{R}^2}$ and $\Delta_{\mathbb{R}^{n-r(B)}}$ given as

$$(5.3) \quad \mathcal{H}_\lambda^{\mathbb{R}^2} = \left(\frac{1}{i} \frac{\partial}{\partial x} \right)^2 + \left(\frac{1}{i} \frac{\partial}{\partial y} + \lambda x \right)^2 \quad \text{and} \quad \Delta_{\mathbb{R}^{n-r(B)}} = \sum_{j=1}^{n-r(B)} \left(\frac{1}{i} \frac{\partial}{\partial x_j} \right)^2$$

on their initial domains $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ and $C_0^\infty(\mathbb{R}^{n-r(B)}) \subset L^2(\mathbb{R}^{n-r(B)})$, respectively. Using Fourier transformation or quasi-modes as in Theorem 26 and Theorem 28 below, the minimal Laplacian $\Delta_{\mathbb{R}^{n-r(B)}}$ is easily seen to have spectrum $\text{spec}(\Delta_{\mathbb{R}^{n-r(B)}}) = [0, \infty)$, see also [Dav95, Theorems 3.5.3, 3.7.4, 8.3.1]. The spectral analysis of $\mathcal{H}_\lambda^{\mathbb{R}^2}$ dates back to Landau in the 1930s. Using dominated convergence, one can show that $\text{Dom}(\mathcal{H}_\lambda^{\mathbb{R}^2})$ contains the space $\mathcal{S}(\mathbb{R}^2) \subset C^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ of Schwartz functions, and the restriction of $\mathcal{H}_\lambda^{\mathbb{R}^2}$ to $\mathcal{S}(\mathbb{R}^2)$ is given by the differential operator in (5.3). We conjugate by the partial Fourier transformation $\mathcal{F}_y: \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{R}^2)$ given by

$$\mathcal{F}_y u(x, \xi_y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-i\xi_y y} dy$$

to obtain

$$(5.4) \quad \mathcal{F}_y \mathcal{H}_\lambda^{\mathbb{R}^2} \mathcal{F}_y^{-1} = -\frac{\partial^2}{\partial x^2} + \lambda^2 \left(x + \frac{\xi_y}{\lambda} \right)^2.$$

The shift $x \rightarrow x + \frac{\xi_y}{\lambda}$ leads to the harmonic oscillator in x with frequency $|\lambda|$, which has the well-known discrete spectrum $\{|\lambda|(2k+1) \mid k = 0, 1, 2, \dots\}$ with eigenfunctions [RS80]

$$u_k(x) = \frac{1}{\sqrt{2^k k!}} \left(\frac{|\lambda|}{\pi} \right)^{\frac{1}{4}} P_k \left(\sqrt{|\lambda|} x \right) e^{-\frac{|\lambda|x^2}{2}}, \quad \text{where} \quad P_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$$

is the k th Hermite polynomial. Each u_k leads to an infinite-dimensional eigenspace of (5.4) containing products of the form $u_k(x + \frac{\xi y}{\lambda})\psi(\xi y)$ where $\psi \in C_0^\infty(\mathbb{R})$. \square

5.2. Higher genus surfaces and $PSL(2, \mathbb{R})$. Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} \simeq \mathbb{R}^2 \mid \text{Im} z = y > 0\}$ denote the upper half plane with its usual metric $ds^2 = y^{-2}(dx^2 + dy^2)$ of constant curvature -1 . Recall that $PSL(2, \mathbb{R})$ can be regarded as the group of orientation-preserving isometries of \mathbb{H} , that is, Möbius transformations

$$z \mapsto \frac{az + b}{cz + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

This allows to view $PSL(2, \mathbb{R})$ as the unit sphere bundle $S\mathbb{H}$ by identifying $(x, y, v) \in S\mathbb{H}$ with the unique Möbius transformation that takes i to $x + iy$, and whose derivative takes $(0, 1) \in T_i\mathbb{H}$ to $v \in T_{x+iy}\mathbb{H}$. We let $\Lambda \subset PSL(2, \mathbb{R})$ be a cocompact lattice that acts on \mathbb{H} without fixed points. The quotient space $M = \Lambda \backslash \mathbb{H}$ is a compact hyperbolic surface, whose unit sphere bundle SM can be identified with $\Lambda \backslash PSL(2, \mathbb{R})$. Following [CFP10, Section 5.2], we let $\mathbf{B} \in \mathbb{R}$ and consider the magnetic flow $\phi^{\mathbf{B}}$ on TM generated by the magnetic field $\mathbf{B} d\text{vol}_M$. In other words, $\phi^{\mathbf{B}}$ is the Hamiltonian flow of $E: TM \rightarrow \mathbb{R}$ given by $E(x, v) = \frac{1}{2}|v|_x^2$ with respect to the twisted symplectic form

$$\omega_{\mathbf{B}} = -\flat^* d\lambda + \mathbf{B} \pi_{TM}^* d\text{vol}_M,$$

where $\flat: TM \rightarrow T^*M$ is the canonical isomorphism induced by the metric, λ is the Liouville 1-form on T^*M , and $\pi_{TM}: TM \rightarrow M$ is the canonical projection. As the lift of $d\text{vol}_M$ to \mathbb{H} has the primitive $\alpha = y^{-1}dx$ with constant norm 1, the corresponding Hamiltonian

$$(5.5) \quad H_{\mathbf{B}}^{\mathbb{H}}: T^*\mathbb{H} \rightarrow \mathbb{R} \quad \text{given by} \quad H_{\mathbf{B}}^{\mathbb{H}}(x, y, p_x, p_y) = \frac{1}{2}((yp_x + \mathbf{B})^2 + y^2 p_y^2)$$

has critical value $\mathbf{c}(H_{\mathbf{B}}^{\mathbb{H}}) \leq \frac{1}{2}\mathbf{B}^2$. In fact, $\mathbf{c}(H_{\mathbf{B}}^{\mathbb{H}}) = \frac{1}{2}\mathbf{B}^2$ as was shown in [Con01, Example 6.2] and [CFP10, Lemma 6.11]. For $k \geq 0$, we denote the restriction of $\phi^{\mathbf{B}}$ to $E^{-1}(k)$ by $\phi^{\mathbf{B}, k}$. Using a simple scaling in the fibres of $T\mathbb{H}$ of the form $(x, v) \mapsto (x, \mathbf{B}v)$, we obtain the following from [CFP10]:

- For $k > \mathbf{c}(H_{\mathbf{B}}^{\mathbb{H}})$, the flow $\phi^{\mathbf{B}, k}$ is conjugate to the underlying geodesic flow on M up to a constant time scaling.
- For $k = \mathbf{c}(H_{\mathbf{B}}^{\mathbb{H}})$, the flow $\phi^{\mathbf{B}, k}$ is conjugate to the horocycle flow on SM up to a constant time scaling, in particular, it has no closed orbits.
- For $0 \leq k < \mathbf{c}(H_{\mathbf{B}}^{\mathbb{H}})$, all orbits of $\phi^{\mathbf{B}, k}$ are closed and their projections to \mathbb{H} are circles.

The quantum analogue of (5.5) is known as the Maass Laplacian [Maa53]

$$(5.6) \quad \mathcal{H}_{\mathbf{B}}^{\mathbb{H}} = \frac{1}{2} y^2 \left(\left(\frac{1}{i} \frac{\partial}{\partial x} + \mathbf{B} y^{-1} \right)^2 + \left(\frac{1}{i} \frac{\partial}{\partial y} \right)^2 \right).$$

Due to its close connections to number theory, in particular, to automorphic forms, $\mathcal{H}_{\mathbf{B}}^{\mathbb{H}}$ has been widely studied in both the mathematics and theoretical physics literature [Roe66a, Roe66b, Els73a, Els73b, Els74, CH85, Com87]. Note that $\mathcal{H}_{\mathbf{B}}^{\mathbb{H}}$ and $\mathcal{H}_{-\mathbf{B}}^{\mathbb{H}}$ are conjugate to each other with respect to the isometry $\mathcal{I}: L^2(\mathbb{H}) \rightarrow L^2(\mathbb{H})$ given by $\mathcal{I}u(x, y) = u(-x, y)$. Derivations of the well-known spectrum of (5.6) can be found in [IS03, MT08], which motivated the spectral analysis in the remainder of this section.

Theorem 25. *The closure of (5.6) has spectrum*

$$(5.7) \quad \text{spec}(\mathcal{H}_B^{\mathbb{H}}) = \begin{cases} [\lambda_{0,\text{cont}}(\mathcal{H}_B^{\mathbb{H}}), \infty) & \text{if } |B| \leq \frac{1}{2} \\ \text{spec}_{\text{pp}}(\mathcal{H}_B^{\mathbb{H}}) \cup [\lambda_{0,\text{cont}}(\mathcal{H}_B^{\mathbb{H}}), \infty) & \text{if } |B| > \frac{1}{2}, \end{cases}$$

where $\lambda_{0,\text{cont}}(\mathcal{H}_B^{\mathbb{H}}) = \frac{1}{2}(|B|^2 + \frac{1}{4})$ and

$$\text{spec}_{\text{pp}}(\mathcal{H}_B^{\mathbb{H}}) = \bigcup_{0 \leq k < |B| - \frac{1}{2}} \left\{ \frac{1}{2}((2k+1)|B| - k(k+1)) \right\} \subset \text{spec}_{\text{ess}}(\mathcal{H}_B^{\mathbb{H}}).$$

The preceding theorem is visualized in Figure 5.1a. We briefly verify that for $|B| > \frac{1}{2}$, any $v \in C_0^\infty((-\infty, 0), \mathbb{C})$ leads to a ground state of (5.6) given by

$$u(x, y) = y^{|B|} \int_{-\infty}^{\infty} e^{\xi_x(ix+y)} v(\xi_x) d\xi_x.$$

Note that $w: \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{C}$ with $w(x, y) = y^{-1}u(x, y)$ is the inverse partial Fourier transform of $\widehat{w}(\xi_x, y) = \sqrt{2\pi} y^{|B|-1} e^{\xi_x y} v(\xi_x)$. We apply Parseval's theorem to obtain

$$\|u\|_{L^2(\mathbb{H})} = \|w\|_{L^2(\mathbb{R} \times \mathbb{R}_+)} = \|\widehat{w}\|_{L^2(\mathbb{R} \times \mathbb{R}_+)},$$

where $\mathbb{R} \times \mathbb{R}_+$ is equipped with standard Lebesgue product measure. Hence,

$$\|u\|_{L^2(\mathbb{H})}^2 \leq 2\pi \int_{-\infty}^{\infty} \int_0^{\infty} y^{2|B|-2} e^{2\xi_x y} |v(\xi_x)|^2 dy d\xi_x < \infty$$

since $|B| > \frac{1}{2}$ and since v vanishes on $[-\varepsilon, \infty)$ for some $\varepsilon > 0$. Moreover,

$$\begin{aligned} \left(y \frac{1}{i} \frac{\partial}{\partial x} + |B| \right)^2 u &= (y\xi_x + |B|)^2 u \\ y^2 \left(\frac{1}{i} \frac{\partial}{\partial y} \right)^2 u &= - (y^2 \xi_x^2 + 2|B|y\xi_x + |B|(|B| - 1)) u. \end{aligned}$$

In the following, we examine the 3-dimensional unit sphere bundle $SM \simeq \Lambda \backslash PSL(2, \mathbb{R})$. The cover $PSL(2, \mathbb{R}) \simeq S\mathbb{H}$ is diffeomorphic to $\mathbb{H} \times \mathbb{S}^1$ with coordinates (x, y, φ) , with respect to which we have the following left-invariant 1-forms [CFP10, Section 6.3]

$$\alpha_1 = \frac{\cos \varphi dx + \sin \varphi dy}{y} \quad \alpha_2 = \frac{-\sin \varphi dx + \cos \varphi dy}{y} \quad \alpha_3 = \frac{dx}{y} + d\varphi.$$

We endow $S\mathbb{H}$ with the left-invariant metric

$$ds^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \frac{1}{y^2} (dx^2 + dy^2 + (y d\varphi + dx)^2),$$

that is,

$$g = \frac{1}{y^2} \begin{pmatrix} 2 & 0 & y \\ 0 & 1 & 0 \\ y & 0 & y^2 \end{pmatrix} \quad \sqrt{|g|} = y^{-2} \quad g^{-1} = \begin{pmatrix} y^2 & 0 & -y \\ 0 & y^2 & 0 \\ -y & 0 & 2 \end{pmatrix}.$$

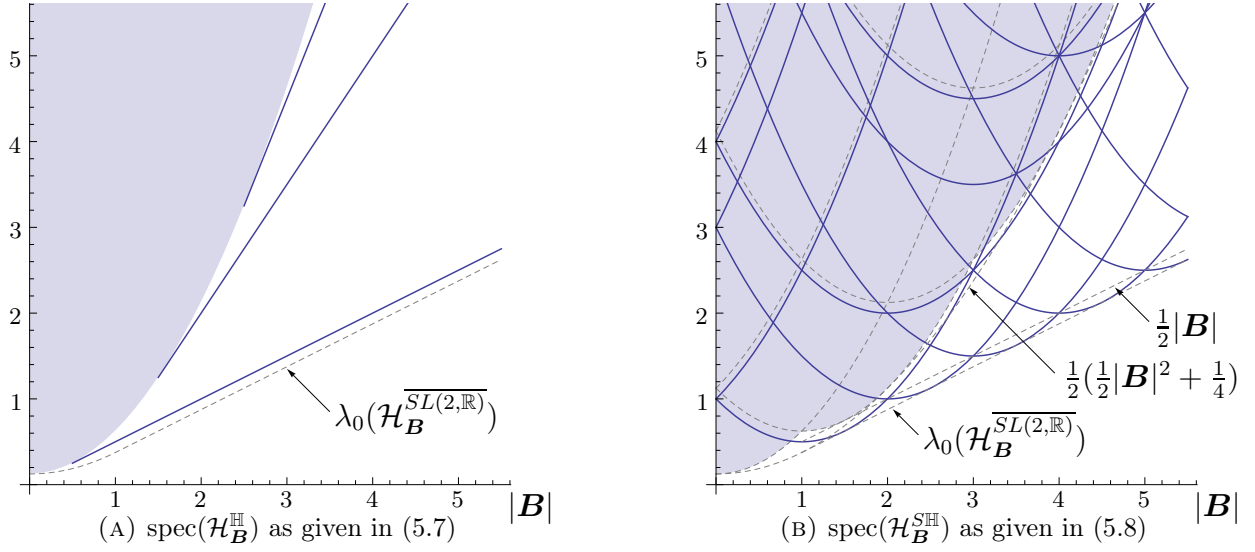


FIGURE 5.1. Plots of (5.7) and (5.8), where solid lines indicate point spectrum and shaded regions indicate continuous spectrum. The dotted lines are drawn for comparison.

We let $\mathbf{B} \in \mathbb{R}$ and consider $\mathbf{B} \alpha_3$ as a left-invariant magnetic potential with associated magnetic field $\beta = \mathbf{B} \frac{1}{y^2} dx \wedge dy$. The corresponding Hamiltonian $H_B^{SM}: T^*SM \rightarrow \mathbb{R}$ and its lift

$$H_B^{\overline{SL(2, \mathbb{R})}}(x, y, \varphi, p_x, p_y, p_\varphi) = \frac{1}{2} ((y p_x - p_\varphi)^2 + (y p_y)^2 + (p_\varphi + \mathbf{B})^2)$$

to the universal cover $\overline{SL(2, \mathbb{R})}$ are known to have critical values [CFP10, Section 6.3]

$$c(H_B^{SM}) = \frac{1}{2} \mathbf{B}^2 \quad \text{and} \quad c(H_B^{\overline{SL(2, \mathbb{R})}}) = \frac{1}{4} \mathbf{B}^2.$$

We examine the corresponding magnetic Schrödinger operator

$$\mathcal{H}_B^{SH} = \frac{1}{2} \left(\left(y \frac{1}{i} \frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial \varphi} \right)^2 + y^2 \left(\frac{1}{i} \frac{\partial}{\partial y} \right)^2 + \left(\frac{1}{i} \frac{\partial}{\partial \varphi} + \mathbf{B} \right)^2 \right)$$

on the intermediate cover $S\mathbb{H}$ and its lift $\mathcal{H}_B^{\overline{SL(2, \mathbb{R})}}$ to $\overline{SL(2, \mathbb{R})}$. Figure 5.1b summarizes the content of the following theorem, which uses the notation $\lfloor |\mathbf{B}| \rfloor = \max\{m \in \mathbb{Z} \mid m \leq |\mathbf{B}|\}$.

Theorem 26. *The closures of \mathcal{H}_B^{SH} and $\mathcal{H}_B^{\overline{SL(2, \mathbb{R})}}$ have spectra*

$$(5.8) \quad \text{spec}(\mathcal{H}_B^{SH}) = \text{spec}_{\text{pp}}(\mathcal{H}_B^{SH}) \cup [\lambda_{0, \text{cont}}(\mathcal{H}_B^{SH}), \infty)$$

$$(5.9) \quad \text{spec}(\mathcal{H}_B^{\overline{SL(2, \mathbb{R})}}) = \text{spec}_{\text{cont}}(\mathcal{H}_B^{\overline{SL(2, \mathbb{R})}}) = [\lambda_0(\mathcal{H}_B^{\overline{SL(2, \mathbb{R})}}), \infty),$$

where

$$\begin{aligned} \text{spec}_{\text{pp}}(\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}) &= \bigcup_{m \in \mathbb{Z} \setminus \{0\}} \bigcup_{0 \leq k < |m|} \left\{ \frac{1}{2} ((\mathbf{B} + m)^2 + (2k+1)|m| - k(k+1)) \right\} \subset \text{spec}_{\text{ess}}(\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}) \\ \lambda_{0,\text{cont}}(\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}) &= \frac{1}{2} \left(\lfloor \|\mathbf{B}\| \rfloor^2 + (\|\mathbf{B}\| - \lfloor \|\mathbf{B}\| \rfloor)^2 + \frac{1}{4} \right) \\ \lambda_0(\mathcal{H}_{\mathbf{B}}^{\overline{SL(2,\mathbb{R})}}) &= \begin{cases} \frac{1}{2} (\frac{1}{2} \|\mathbf{B}\|^2 + \frac{1}{4}) & \text{if } \|\mathbf{B}\| \leq 1 \\ \frac{1}{2} (\|\mathbf{B}\| - \frac{1}{4}) & \text{if } \|\mathbf{B}\| > 1. \end{cases} \end{aligned}$$

In particular,

$$\lambda_0(\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}) = \begin{cases} \frac{1}{2} (\|\mathbf{B}\|^2 + \frac{1}{4}) & \text{if } \|\mathbf{B}\| \leq \frac{7}{8} \\ \frac{1}{2} (1 + (1 - \|\mathbf{B}\|)^2) & \text{if } \frac{7}{8} < \|\mathbf{B}\| \leq 1 \\ \frac{1}{2} (\lfloor \|\mathbf{B}\| \rfloor + (\|\mathbf{B}\| - \lfloor \|\mathbf{B}\| \rfloor)^2) & \text{if } \|\mathbf{B}\| > 1, \end{cases}$$

which has local minima at $\|\mathbf{B}\| = \frac{7}{8}$, and $\frac{1}{2}(\|\mathbf{B}\| - \frac{1}{4}) \leq \lambda_0(\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}) \leq \frac{1}{2}\|\mathbf{B}\|$ for $\|\mathbf{B}\| \geq 1$.

Proof. The operators $\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}$ and $\mathcal{H}_{\mathbf{B}}^{\overline{SL(2,\mathbb{R})}}$ allow for a partial diagonalization by means of discrete and continuous Fourier transformation, respectively. Since we use both techniques repeatedly in the following sections, we recall them in detail.

Let $D \subset \text{Dom}(\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}})$ denote the set of functions which are finite linear combinations of products uw with $u \in C_0^\infty(\mathbb{H})$ and $w \in C^\infty(\mathbb{S}^1)$. According to [RS75, Theorem II.10], the mapping $u \otimes w \mapsto uw$ from $L^2(\mathbb{H}, y^{-2} dx dy) \otimes L^2(\mathbb{S}^1, d\varphi)$ to $L^2(S\mathbb{H}) \simeq L^2(\mathbb{H} \times \mathbb{S}^1, y^{-2} dx dy d\varphi)$ extends to an isometry, which implies that D is dense in $L^2(S\mathbb{H})$. Fourier analysis on \mathbb{S}^1 shows that the functions $(e_m)_{m \in \mathbb{Z}}$ given by $e_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$ yield a complete orthonormal set of eigenfunctions of the symmetric operator $\frac{1}{i} \frac{\partial}{\partial \varphi}$. Thus, $L^2(\mathbb{S}^1, d\varphi) \simeq \bigoplus_{m \in \mathbb{Z}} \mathbb{C} e_m$, and therefore

$$L^2(S\mathbb{H}) \simeq \bigoplus_{m=-\infty}^{\infty} L_m \quad \text{with} \quad L_m = L^2(\mathbb{H}) \otimes \mathbb{C} e_m.$$

The number m is the quantum analogue of the classical angular momentum. On each $D_m = D \cap L_m$, $\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}$ reduces to a shifted version of the Maass Laplacian (5.6), namely,

$$\mathcal{H}_{\mathbf{B},m}^{S\mathbb{H}} = \frac{1}{2} \left(\left(y \frac{1}{i} \frac{\partial}{\partial x} - m \right)^2 + y^2 \left(\frac{1}{i} \frac{\partial}{\partial y} \right)^2 + (m + \mathbf{B})^2 \right) = \mathcal{H}_{-m}^{\mathbb{H}} + \frac{1}{2} (m + \mathbf{B})^2$$

with domain $C_0^\infty(\mathbb{H})$. From (5.7), we get

$$\text{spec}(\mathcal{H}_{\mathbf{B},m}^{S\mathbb{H}}) = \begin{cases} [\frac{1}{2} (\mathbf{B}^2 + \frac{1}{4}), \infty) & \text{if } m = 0 \\ \text{spec}_{\text{pp}}(\mathcal{H}_{\mathbf{B},m}^{S\mathbb{H}}) \cup [\frac{1}{2} ((\mathbf{B} + m)^2 + |m|^2 + \frac{1}{4}), \infty) & \text{if } m \neq 0, \end{cases}$$

where

$$\text{spec}_{\text{pp}}(\mathcal{H}_{\mathbf{B},m}^{S\mathbb{H}}) = \bigcup_{0 \leq k < |m|} \left\{ \frac{1}{2} ((\mathbf{B} + m)^2 + (2k+1)|m| - k(k+1)) \right\} \subset \text{spec}_{\text{ess}}(\mathcal{H}_{\mathbf{B},m}^{S\mathbb{H}}).$$

The claim (5.8) now follows from

$$\text{spec}(\mathcal{H}_{\mathbf{B}}^{S\mathbb{H}}) = \overline{\bigcup_{m \in \mathbb{Z}} \text{spec}(\mathcal{H}_{\mathbf{B},m}^{S\mathbb{H}})}.$$

In order to determine $\text{spec}(\overline{\mathcal{H}_{\mathbf{B}}^{SL(2, \mathbb{R})}})$, we use that

$$L^2(\overline{SL(2, \mathbb{R})}) \simeq L^2(\mathbb{H} \times \mathbb{R}, y^{-2} dx dy d\varphi)$$

and consider the partial Fourier transformation

$$\mathcal{F}_\varphi : L^2(\mathbb{H} \times \mathbb{R}, y^{-2} dx dy d\varphi) \rightarrow L^2(\mathbb{H} \times \mathbb{R}, y^{-2} dx dy d\xi_\varphi)$$

given by

$$\mathcal{F}_\varphi u(x, y, \xi_\varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y, \varphi) e^{-i\xi_\varphi \varphi} d\varphi.$$

Let D denote the linear span of $C_0^\infty(\mathbb{H}) \times \mathcal{S}(\mathbb{R})$. Using cut-off functions, dominated convergence and the canonical isometry $L^2(\mathbb{H} \times \mathbb{R}) \simeq L^2(\mathbb{H}) \otimes L^2(\mathbb{S}^1)$ [RS75, Theorem II.10], one sees that D is a core for $\overline{\mathcal{H}_{\mathbf{B}}^{SL(2, \mathbb{R})}}$. Moreover, \mathcal{F}_φ restricts to an isometry of D , on which we have

$$(5.10) \quad \mathcal{F}_\varphi \overline{\mathcal{H}_{\mathbf{B}}^{SL(2, \mathbb{R})}} \mathcal{F}_\varphi^{-1} = \frac{1}{2} \left(\left(y \frac{1}{i} \frac{\partial}{\partial x} - \xi_\varphi \right)^2 + y^2 \left(\frac{1}{i} \frac{\partial}{\partial y} \right)^2 + (\xi_\varphi + \mathbf{B})^2 \right).$$

More generally, one can use the canonical isomorphism [RS80, Theorem II.10]

$$\mathcal{I} : L^2(\mathbb{H} \times \mathbb{R}, y^{-2} dx dy d\xi_\varphi) \rightarrow L^2(\mathbb{R}, d\xi_\varphi, L^2(\mathbb{H}, y^{-2} dx dy))$$

given by $\mathcal{I}\hat{u}(\xi_\varphi)(x, y) = \hat{u}(x, y, \xi_\varphi)$ to reinterpret (5.10) as a constant fibre direct integral decomposition over $L^2(\mathbb{R}, d\xi_\varphi, L^2(\mathbb{H}, y^{-2} dx dy))$ [RS78, Section XIII.16]. For each $\xi_\varphi \in \mathbb{R}$, we define $\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}$ as the differential operator (5.10) acting on $C_0^\infty(\mathbb{H}) \subset L^2(\mathbb{H}, y^{-2} dx dy d\xi_\varphi)$. According to (5.7), we have

$$\text{spec}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}) = \begin{cases} \left[\lambda_{0, \text{cont}}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}), \infty \right) & \text{if } |\xi_\varphi| \leq \frac{1}{2} \\ \text{spec}_{\text{pp}}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}) \cup \left[\lambda_{0, \text{cont}}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}), \infty \right) & \text{if } |\xi_\varphi| > \frac{1}{2}, \end{cases}$$

where

$$\lambda_{0, \text{cont}}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}) = \frac{1}{2} \left((\mathbf{B} + \xi_\varphi)^2 + \xi_\varphi^2 + \frac{1}{4} \right) = \left(\frac{1}{2} \mathbf{B} + \xi_\varphi \right)^2 + \frac{1}{2} \left(\frac{1}{2} \mathbf{B}^2 + \frac{1}{4} \right)$$

and

$$\text{spec}_{\text{pp}}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}) = \bigcup_{0 \leq k < |\xi_\varphi| - \frac{1}{2}} \left\{ \frac{1}{2} \left((\mathbf{B} + \xi_\varphi)^2 + (2k+1)|\xi_\varphi| - k(k+1) \right) \right\},$$

with infimum $\frac{1}{2}(|\mathbf{B}| - \frac{1}{4})$ for $|\mathbf{B}| > 1$ attained at $k = 0$ and $\xi_\varphi = \frac{\mathbf{B}}{|\mathbf{B}|}(\frac{1}{2} - |\mathbf{B}|)$. Since the mappings $\xi_\varphi \mapsto \lambda_{0, \text{cont}}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}})$ and $\xi_\varphi \mapsto \frac{1}{2}((\mathbf{B} + \xi_\varphi)^2 + (2k+1)|\xi_\varphi| - k(k+1))$ are continuous, we can use [RS78, Theorem XIII.85] to obtain

$$\text{spec}(\overline{\mathcal{H}_{\mathbf{B}}^{SL(2, \mathbb{R})}}) = \bigcup_{\xi_\varphi \in \mathbb{R}} \text{spec}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}}).$$

Moreover, $\lambda \in \text{spec}_{\text{pp}}(\overline{\mathcal{H}_{\mathbf{B}}^{SL(2, \mathbb{R})}})$ if and only if $\{\xi_\varphi \in \mathbb{R} \mid \lambda \in \text{spec}_{\text{pp}}(\overline{\mathcal{H}_{\mathbf{B}, \xi_\varphi}^{SL(2, \mathbb{R})}})\}$ has non-zero Lebesgue measure. Thus, $\text{spec}_{\text{pp}}(\overline{\mathcal{H}_{\mathbf{B}}^{SL(2, \mathbb{R})}}) = \emptyset$, which completes the proof. \square

It is worth mentioning, that Erdős [Erd97, Section E, Remark 1] constructed a radially symmetric magnetic potential α on the Euclidean space \mathbb{R}^n such that $\lambda_0(\mathbf{B}\alpha)$ is a non-monotone function of \mathbf{B} . However, α has non-constant norm whereas α_3 above is left-invariant. The spectrum of the Laplacian $\Delta^{S\mathbb{H}} = 2\mathcal{H}_0^{S\mathbb{H}}$ should be compared with [Sun88, Example A].

5.3. Heisenberg group. The Heisenberg group Nil is the semidirect product $\mathbb{R} \ltimes_{\eta} \mathbb{R}^2$ with $\eta: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^2)$ given by $\eta(x)(y, z) = (y, xy + z)$. In other words, Nil is \mathbb{R}^3 viewed as a nilpotent Lie group with multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$$

coming from the matrix representation

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Following [CFP10], we consider the cocompact lattice $\Lambda = \mathbb{Z} \ltimes_{\eta} \mathbb{Z}^2$ of matrices with $x, y, z \in \mathbb{Z}$. All cocompact lattices of Nil are isomorphic to Λ [Sco83]. The left-invariant 1-forms

$$\alpha_1 = dx \quad \alpha_2 = dy \quad \alpha_3 = dz - x dy$$

give rise to the left-invariant metric

$$ds^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = dx^2 + dy^2 + (dz - x dy)^2,$$

that is,

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{pmatrix} \quad \sqrt{|g|} = 1 \quad g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1+x^2 \end{pmatrix}.$$

We let $\mathbf{B} \in \mathbb{R}$ and regard $\mathbf{B}\alpha_3$ as a left-invariant magnetic potential. The corresponding Hamiltonian $H_{\mathbf{B}}^{\Lambda \backslash \text{Nil}}: T^*(\Lambda \backslash \text{Nil}) \rightarrow \mathbb{R}$ and its lift $H_{\mathbf{B}}^{\text{Nil}}: T^*\text{Nil} \rightarrow \mathbb{R}$ given by

$$H_{\mathbf{B}}^{\text{Nil}}(x, y, z, p_x, p_y, p_z) = \frac{1}{2} (p_x^2 + ((p_y - \mathbf{B}x) + x(p_z + \mathbf{B}))^2 + (p_z + \mathbf{B})^2)$$

have identical critical values [CFP10, Section 6.3]

$$c(H_{\mathbf{B}}^{\Lambda \backslash \text{Nil}}) = c(H_{\mathbf{B}}^{\text{Nil}}) = \frac{1}{2} \mathbf{B}^2$$

since $\pi_1(\Lambda \backslash \text{Nil}) \simeq \Lambda$ is nilpotent and therefore amenable. The maximal abelian cover of $\Lambda \backslash \text{Nil}$ is $[\Lambda, \Lambda] \backslash \text{Nil}$, where $[\Lambda, \Lambda] = \{0\} \times \{0\} \times \mathbb{Z}$. In the following, we study the corresponding magnetic Schrödinger operator $\mathcal{H}_{\mathbf{B}}^{[\Lambda, \Lambda] \backslash \text{Nil}}$ acting as

$$\mathcal{H}_{\mathbf{B}}^{[\Lambda, \Lambda] \backslash \text{Nil}} = \frac{1}{2} \left(\left(\frac{1}{i} \frac{\partial}{\partial x} \right)^2 + \left(\frac{1}{i} \frac{\partial}{\partial y} + x \frac{1}{i} \frac{\partial}{\partial z} \right)^2 + \left(\frac{1}{i} \frac{\partial}{\partial z} + \mathbf{B} \right)^2 \right)$$

on $C_0^\infty(\mathbb{R}^2 \times \mathbb{Z} \backslash \mathbb{R}) \subset L^2(\mathbb{R}^2 \times \mathbb{Z} \backslash \mathbb{R}) \simeq L^2([\Lambda, \Lambda] \backslash \text{Nil})$, and its lift $\mathcal{H}_{\mathbf{B}}^{\text{Nil}}$ to Nil with domain $C_0^\infty(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \simeq L^2(\text{Nil})$. Figure 5.2 visualizes the following theorem.

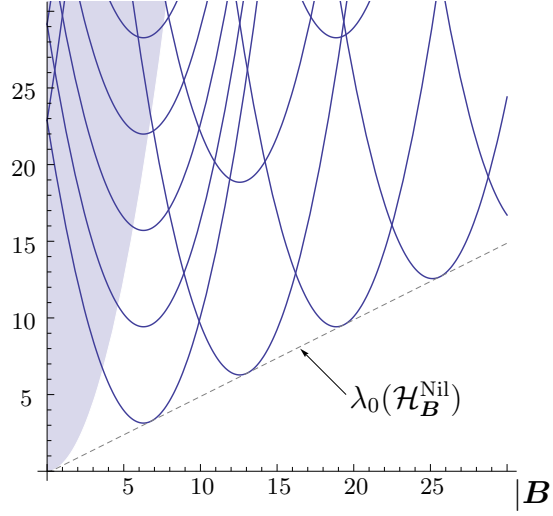


FIGURE 5.2. Plot of (5.11), where solid lines indicate point spectrum and the shaded region indicates continuous spectrum.

Theorem 27. *The closures of $\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}}$ and $\mathcal{H}_B^{\text{Nil}}$ have spectra*

$$(5.11) \quad \text{spec}(\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}}) = \text{spec}_{\text{pp}}(\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}}) \cup [\lambda_{0, \text{cont}}(\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}}), \infty)$$

$$(5.12) \quad \text{spec}(\mathcal{H}_B^{\text{Nil}}) = \text{spec}_{\text{cont}}(\mathcal{H}_B^{\text{Nil}}) = [\lambda_0(\mathcal{H}_B^{\text{Nil}}), \infty),$$

where

$$\text{spec}_{\text{pp}}(\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}}) = \bigcup_{m \in \mathbb{Z} \setminus \{0\}} \bigcup_{k \in \mathbb{N}_0} \left\{ \frac{1}{2} ((B + 2\pi m)^2 + 2\pi(2k+1)|m|) \right\} \subset \text{spec}_{\text{ess}}(\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}})$$

$$\lambda_{0, \text{cont}}(\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}}) = \frac{1}{2}|B|^2$$

$$\lambda_0(\mathcal{H}_B^{\text{Nil}}) = \begin{cases} \frac{1}{2}|B|^2 & \text{if } |B| \leq \frac{1}{2} \\ \frac{1}{2}(|B| - \frac{1}{4}) & \text{if } |B| > \frac{1}{2}. \end{cases}$$

In particular, the function $B \mapsto \lambda_0(\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}})$ has countably many local minima.

Proof. We mimic the proof of Theorem 26 and use that $\mathcal{H}_B^{[\Lambda, \Lambda] \setminus \text{Nil}}$ and $\mathcal{H}_B^{\text{Nil}}$ allow for discrete and continuous Fourier transformation in the z -coordinate, respectively. One obtains shifted versions of the magnetic Schrödinger operators in (5.3) acting on $C_0^\infty(\mathbb{R}^2)$ as

$$\mathcal{H}_{B, \xi_z}^{[\Lambda, \Lambda] \setminus \text{Nil}} = \mathcal{H}_{B, \xi_z}^{\text{Nil}} = \frac{1}{2} \left(\left(\frac{1}{i} \frac{\partial}{\partial x} \right)^2 + \left(\frac{1}{i} \frac{\partial}{\partial y} + 2\pi \xi_z x \right)^2 + (2\pi \xi_z + B)^2 \right)$$

with $\xi_z \in \mathbb{Z}$ in the former case and $\xi_z \in \mathbb{R}$ in the latter case. According to Theorem 24,

$$\text{spec}(\mathcal{H}_{B, \xi_z}^{\text{Nil}}) = \begin{cases} [\frac{1}{2}B^2, \infty) & \text{if } \xi_z = 0 \\ \left\{ \frac{1}{2}((B + 2\pi \xi_z)^2 + 2\pi(2k+1)|\xi_z|) \mid k = 0, 1, 2, \dots \right\} & \text{if } \xi_z \neq 0, \end{cases}$$

with ground state energy

$$(5.13) \quad \lambda_0(\mathcal{H}_{B, \xi_z}^{\text{Nil}}) = \frac{1}{2}((B + 2\pi \xi_z)^2 + 2\pi|\xi_z|).$$

Using

$$\operatorname{spec}(\mathcal{H}_{\mathbf{B}}^{[\Lambda, \Lambda] \setminus \text{Nil}}) = \bigcup_{m \in \mathbb{Z}} \operatorname{spec}(\mathcal{H}_{\mathbf{B}, m}^{[\Lambda, \Lambda] \setminus \text{Nil}}) = \bigcup_{m \in \mathbb{Z}} \operatorname{spec}(\mathcal{H}_{\mathbf{B}, m}^{\text{Nil}}),$$

we easily deduce (5.11). As for (5.12), note that $\mathcal{H}_{\mathbf{B}}^{\text{Nil}}$ is conjugate to a direct integral of model operators $\mathcal{H}_{\mathbf{B}, \xi_z}^{\text{Nil}}$ over $L^2(\mathbb{R}, d\xi_z, L^2(\mathbb{R}^2, dx dy))$. On the full measure set $\{\xi_z \neq 0\} = \mathbb{R} \setminus \{0\} \subset \mathbb{R}$, the operators $\mathcal{H}_{\mathbf{B}, \xi_z}^{\text{Nil}}$ have pure point spectrum with nowhere constant eigenvalue functions that depend continuously on ξ_z , hence [RS78, Theorem XIII.85]

$$\operatorname{spec}(\mathcal{H}_{\mathbf{B}}^{\text{Nil}}) = \bigcup_{\xi_z \neq 0} \operatorname{spec}(\mathcal{H}_{\mathbf{B}, \xi_z}^{\text{Nil}}),$$

and $\operatorname{spec}_{\text{pp}}(\mathcal{H}_{\mathbf{B}}^{\text{Nil}}) = \emptyset$. The claim now follows by an inspection of (5.13), namely,

$$\inf_{\xi_z \neq 0} \lambda_0(\mathcal{H}_{\mathbf{B}, \xi_z}^{\text{Nil}}) = \begin{cases} \frac{1}{2} |\mathbf{B}|^2 & \text{obtained for } \xi_z \rightarrow 0 \\ \frac{1}{2} (|\mathbf{B}| - \frac{1}{4}) & \text{attained at } \xi_z = \frac{1}{2\pi} \frac{\mathbf{B}}{|\mathbf{B}|} (|\mathbf{B}| - \frac{1}{4}) \end{cases} \quad \begin{array}{l} \text{if } |\mathbf{B}| \leq \frac{1}{2} \\ \text{if } |\mathbf{B}| > \frac{1}{2} \end{array}.$$

□

5.4. Solvable geometry. The Lie group Sol is the semidirect product $\text{Sol} = \mathbb{R}^2 \rtimes_{\eta} \mathbb{R}$ with $\eta: \mathbb{R} \rightarrow \operatorname{Aut}(\mathbb{R}^2)$ given by $\eta(z)(x, y) = (e^z x, e^{-z} y)$. In other words, Sol is the manifold \mathbb{R}^3 equipped with the multiplication

$$(x, y, z)(x', y', z') = (x + e^z x', y + e^{-z} y', z + z')$$

coming from the matrix representation

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The left-invariant 1-forms

$$\alpha_x = e^{-z} dx \quad \alpha_y = e^z dy \quad \alpha_z = dz$$

give rise to the left-invariant metric

$$ds^2 = \alpha_x^2 + \alpha_y^2 + \alpha_z^2 = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2,$$

in particular, $L^2(\text{Sol}) = L^2(\mathbb{R}^3, dx dy dz)$. Following [BP08], we consider compact quotients of Sol obtained from hyperbolic gluing maps of \mathbb{T}^2 as follows. Let $A \in SL(2, \mathbb{Z})$ have real eigenvalues $\lambda > 1$ and $\lambda^{-1} < 1$, and let $P \in GL(2, \mathbb{R})$ be such that

$$(5.14) \quad PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

The image of the injective homomorphism

$$(5.15) \quad \mathbb{Z}^2 \rtimes_A \mathbb{Z} \hookrightarrow \text{Sol} \quad \text{given by} \quad \left(\begin{pmatrix} m \\ n \end{pmatrix}, l \right) \mapsto \left(P \begin{pmatrix} m \\ n \end{pmatrix}, l \log \lambda \right)$$

is a cocompact lattice Λ_A in Sol. The closed 3-manifold $\Lambda_A \setminus \text{Sol}$ is a torus bundle over \mathbb{S}^1 . Its harmonic monopole $dx \wedge dy$ generates $H^2(\Lambda_A \setminus \text{Sol}, \mathbb{R})$ and is Hodge dual to the generator dz of $H^1(\Lambda_A \setminus \text{Sol}, \mathbb{R})$. For the sake of convenience, we introduce

$$\mathcal{D}_x = \frac{1}{i} \frac{\partial}{\partial x} \quad \mathcal{D}_y = \frac{1}{i} \frac{\partial}{\partial y} \quad \mathcal{D}_z = \frac{1}{i} \frac{\partial}{\partial z}.$$

A simple integration by parts shows that \mathcal{D}_x , \mathcal{D}_y and \mathcal{D}_z are symmetric on $C_0^\infty(\mathbb{R}^3)$, on which we have $\Delta^{\text{Sol}} = e^{2z} \mathcal{D}_x^2 + e^{-2z} \mathcal{D}_y^2 + \mathcal{D}_z^2$. Since Sol is a globally symmetric space of non-compact type, $\text{spec}(\Delta^{\text{Sol}})$ is known to be purely continuous and of the form $[\lambda_0(\Delta^{\text{Sol}}), \infty)$ for some $\lambda_0(\Delta^{\text{Sol}}) \geq 0$, see also [Sun88]. On the other hand, Sol is simply connected and has cocompact, solvable and therefore amenable lattices, which yields $\lambda_0(\Delta^{\text{Sol}}) = 0$ by virtue of Brooks' theorem [Bro81, Theorem 1]. Using ideas from [IS03], we give a more basic derivation of $\text{spec}(\Delta^{\text{Sol}}) = \text{spec}_{\text{cont}}(\Delta^{\text{Sol}}) = [0, \infty)$ in the following section.

5.4.1. Exact case. Let $(\mathbf{B}_x, \mathbf{B}_y) \in \mathbb{R}^2$ and denote its norm by $\mathbf{B} = \sqrt{\mathbf{B}_x^2 + \mathbf{B}_y^2}$. We consider the left-invariant magnetic potential $\mathbf{B}_x \alpha_x + \mathbf{B}_y \alpha_y = \mathbf{B}_x e^{-z} dx + \mathbf{B}_y e^z dy$ with associated Hamiltonian

$$H_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}(x, y, z, p_x, p_y, p_z) = \frac{1}{2} \left((e^z p_x + \mathbf{B}_x)^2 + (e^{-z} p_y + \mathbf{B}_y)^2 + p_z^2 \right).$$

Note that $H_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}$ descends to any quotient of Sol. Using the same arguments as in [MP10, Section 3.1], one sees that any cocompact lattice $\Lambda_A \subset \text{Sol}$ as in (5.15) satisfies

$$\mathbf{c}(H_{\mathbf{B}_x, \mathbf{B}_y}^{\Lambda_A \setminus \text{Sol}}) = \mathbf{c}(H_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}) = \frac{1}{2} \mathbf{B}^2.$$

For the case $\mathbf{B}_y = 0$, Macarini and Schlenk [MS11, Proposition 7.1] discovered that the magnetic flow on $(H_{\mathbf{B}_x, 0}^{\Lambda_A \setminus \text{Sol}})^{-1}(k)$ has non-vanishing topological entropy if and only if $k > \mathbf{c}(H_{\mathbf{B}_x, 0}^{\text{Sol}})$. For $\mathbf{B} \leq \frac{1}{2}$, the critical energy $\mathbf{c}(H_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}})$ turns out to coincide with the ground state energy of the corresponding magnetic Schrödinger operator, which is given as

$$(5.16) \quad \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} = \frac{1}{2} \left((e^z \mathcal{D}_x + \mathbf{B}_x)^2 + (e^{-z} \mathcal{D}_y + \mathbf{B}_y)^2 + \mathcal{D}_z^2 \right)$$

on its initial domain $C_0^\infty(\mathbb{R}^3)$.

Theorem 28. *The spectrum of $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}$ depends on $\mathbf{B} = \sqrt{\mathbf{B}_x^2 + \mathbf{B}_y^2}$ as follows:*

- (1) *We have $\text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}) = \text{spec}(\mathcal{H}_{\mathbf{B}_y, \mathbf{B}_x}^{\text{Sol}}) \supseteq [\frac{1}{2} \mathbf{B}^2, \infty)$.*
- (2) *If $0 \leq \mathbf{B} \leq \frac{1}{2}$, then equality holds in 1., that is, $\text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}) = [\frac{1}{2} \mathbf{B}^2, \infty)$.*
- (3) *If $\mathbf{B} > \frac{1}{2}$, then we have $\lambda_0(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}) \geq \frac{1}{2} (\mathbf{B} - \frac{1}{4})$ with equality if also $\mathbf{B}_y = 0$.*
- (4) *If $|\mathbf{B}_x| > \frac{1}{2}$, then $\frac{1}{2}(|\mathbf{B}_x| + |\mathbf{B}_y|^2 - \frac{1}{4}) \in \text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}})$.*

Proof of (1). Since $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}$ and $\mathcal{H}_{\mathbf{B}_y, \mathbf{B}_x}^{\text{Sol}}$ are conjugate to each other with respect to the idempotent isometry $\mathcal{I}: L^2(\text{Sol}) \rightarrow L^2(\text{Sol})$ given by $\mathcal{I}u(x, y, z) = u(y, x, -z)$, their spectra coincide. In order to verify $\text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}) \supseteq [\frac{1}{2} \mathbf{B}^2, \infty)$, it suffices to show that for any $\kappa \geq 0$ there exists a Weyl sequence $(u_n)_{n \in \mathbb{N}}$ with $u_n \in C_0^\infty(\mathbb{R}^3)$ such that

$$\|u_n\|_{L^2(\mathbb{R}^3)} \rightarrow \sqrt{\pi} \quad \text{and} \quad \|(2\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} - \mathbf{B}^2 - \kappa^2)u_n\|_{L^2(\mathbb{R}^3)} \rightarrow 0.$$

We consider products of the form $u_n(x, y, z) = \chi_n(x) \psi_n(y) \zeta_n(z) \gamma_n(x, y)$ with $\chi_n, \psi_n, \zeta_n \in C_0^\infty(\mathbb{R})$ and $\gamma_n \in C^\infty(\mathbb{R}^2)$. The classical Poincaré inequality [Eva10, Section 5.8.1] implies that for each compact set $K \subset \mathbb{R}$ there exists $C_K > 0$ such that if ζ_n is supported inside K , we have

$$\|\zeta_n\|_{L^2(\mathbb{R})} \leq C_K \|\mathcal{D}_z^2 \zeta_n\|_{L^2(\mathbb{R})}.$$

Hence, we choose $(\chi_n, \psi_n, \zeta_n)_{n \in \mathbb{N}}$ with growing supports. In order to motivate the choices below, we conjugate (5.16) by the partial Fourier transformation $\mathcal{F}_{x,y}: L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ given by

$$\mathcal{F}_{x,y}u(\xi_x, \xi_y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, z) e^{-i(\xi_x x + \xi_y y)} dx dy$$

to obtain that on $\mathcal{F}_{x,y}(C_0^\infty(\mathbb{R}^3))$

$$\mathcal{F}_{x,y} \left(2 \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} - \mathbf{B}^2 \right) \mathcal{F}_{x,y}^{-1} = \mathcal{D}_z^2 + \xi_x^2 e^{2z} + 2\mathbf{B}_x \xi_x e^z + \xi_y^2 e^{-2z} + 2\mathbf{B}_y \xi_y e^{-z}.$$

This suggests to choose $(\gamma_n)_{n \in \mathbb{N}}$ such that the transforms $(\mathcal{F}_{x,y} \zeta_n \gamma_n \mathcal{F}_{x,y}^{-1})_{n \in \mathbb{N}}$ concentrate around the line $\{0\} \times \{0\} \times \mathbb{R}$ fast enough to compensate for the growing support of $(\zeta_n)_{n \in \mathbb{N}}$ and the resulting growth of the factors $(e^{kz} \zeta_n)_{n \in \mathbb{N}}$ for $k = -2, -1, 1, 2$. We control these concentration and growth rates by real sequences $(c_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ with $c_n \searrow 0$ and $s_n \nearrow \infty$. First, we choose $(\chi_n)_{n \in \mathbb{N}}$ as smooth real-valued cut-off functions on \mathbb{R} with

$$(5.17) \quad 0 \leq \chi_n \leq 1 \quad \chi_n \equiv 1 \text{ on } [-s_n, s_n] \quad \chi_n \equiv 0 \text{ outside } [-(s_n + 1), (s_n + 1)],$$

and such that the first and second derivatives are uniformly bounded by some $C > 0$, that is,

$$\sup_{n \in \mathbb{N}} \max \left\{ \|\chi_n'\|_{L^\infty(\mathbb{R})}, \|\chi_n''\|_{L^\infty(\mathbb{R})} \right\} \leq C.$$

We let $(\psi_n)_{n \in \mathbb{N}} = (\chi_n)_{n \in \mathbb{N}}$ denote the same sequence in $C_0^\infty(\mathbb{R})$ with the implicit understanding that χ_n and ψ_n will henceforth be regarded as functions of x and y , respectively. Moreover, we choose some $v \in C_0^\infty(\mathbb{R}, \mathbb{R})$ with norm $\|v\|_{L^2(\mathbb{R})} = 1$ and support in $(\frac{1}{2}, 1)$ to define

$$\zeta_n(z) = 2^{-\frac{n}{2}} e^{i\kappa z} v(2^{-n}z), \quad \text{and let} \quad \gamma_n(x, y) = c_n e^{-\frac{1}{2}c_n^2(x^2+y^2)}.$$

Note that

$$u_n(x, y, z) = 2^{-\frac{n}{2}} c_n \chi_n(x) \psi_n(y) e^{i\kappa z} e^{-\frac{1}{2}c_n^2(x^2+y^2)} v(2^{-n}z)$$

is supported in $\mathbb{R}^2 \times (2^{n-1}, 2^n)$, for which reason $(u_n)_{n \in \mathbb{N}}$ is an orthogonal sequence. If we require $c_n s_n \rightarrow \infty$, then the dominated convergence theorem yields

$$(5.18) \quad \begin{aligned} \|u_n\|^2 &= \int_{-(s_n+1)}^{s_n+1} \chi_n^2(x) e^{-c_n^2 x^2} c_n dx \int_{-(s_n+1)}^{s_n+1} \psi_n^2(y) e^{-c_n^2 y^2} c_n dy \int_{-\infty}^{\infty} v^2(2^{-n}z) 2^{-n} dz \\ &= \left(\int_{-c_n(s_n+1)}^{c_n(s_n+1)} \chi_n^2(c_n^{-1}t) e^{-t^2} dt \right)^2 \rightarrow \pi. \end{aligned}$$

One easily computes that

$$(5.19) \quad \mathcal{D}_x^2 u_n(x, y, z) = \psi_n(y) \zeta_n(z) \gamma_n(x, y) \left((c_n^2 - c_n^4 x^2) \chi_n(x) + 2 c_n^2 x \chi_n'(x) - \chi_n''(x) \right).$$

The first summand equals $(c_n^2 - c_n^4 x^2)u_n(x, y, z)$, and we obtain

$$\begin{aligned}
 \|e^{2z}(c_n^2 - c_n^4 x^2)u_n\|^2 &\leq \int_{-\infty}^{\infty} (c_n^2 - c_n^4 x^2)^2 e^{-c_n^2 x^2} c_n dx \int_{-\infty}^{\infty} e^{-c_n^2 y^2} c_n dy \int_{-\infty}^{\infty} e^{4z} v^2(2^{-n}z) 2^{-n} dz \\
 (5.20) \quad &\leq \sqrt{\pi} c_n^4 \int_{-\infty}^{\infty} (1-t^2)^2 e^{-t^2} dt \sup_{z \in [2^{n-1}, 2^n]} |e^{4z}| \leq \frac{3}{4} \pi c_n^4 e^{2^{n+2}}.
 \end{aligned}$$

As for the second summand, note that $\chi_n' \equiv 0$ outside $I_n = [-(s_n + 1), -s_n] \cup [s_n, s_n + 1]$, so

$$\begin{aligned}
 \|c_n^2 x e^{2z} \chi_n' \psi_n \zeta_n \gamma_n\|^2 &\leq c_n^2 e^{2^{n+2}} \int_{I_n} (c_n x \chi_n'(x))^2 e^{-c_n^2 x^2} c_n dx \int_{-\infty}^{\infty} e^{-c_n^2 y^2} c_n dy \\
 (5.21) \quad &\leq 2\sqrt{\pi} C^2 c_n^2 e^{2^{n+2}} \int_{c_n s_n}^{c_n(s_n+1)} t^2 e^{-t^2} dt,
 \end{aligned}$$

and similarly for $e^{2z} \chi_n'' \psi_n \zeta_n \gamma_n$. We set $c_n = e^{-3^n}$ and $s_n = e^{4^n}$ to satisfy $c_n s_n \rightarrow \infty$ and $c_n^2 e^{2^{n+2}} \rightarrow 0$, in particular, $\|e^{2z} \mathcal{D}_x^2 u_n\|_{L^2(\mathbb{R}^3)} \rightarrow 0$. Similar computations show that

$$\|e^z \mathcal{D}_x u_n\|_{L^2(\mathbb{R}^3)} \rightarrow 0 \quad \|e^{-2z} \mathcal{D}_y^2 u_n\|_{L^2(\mathbb{R}^3)} \rightarrow 0 \quad \|e^{-z} \mathcal{D}_y u_n\|_{L^2(\mathbb{R}^3)} \rightarrow 0.$$

Since

$$\mathcal{D}_z \zeta_n(z) = 2^{-\frac{n}{2}} e^{i\kappa z} (\kappa v(2^{-n}z) - 2^{-n} i v'(2^{-n}z)),$$

we have

$$\mathcal{D}_z^2 \zeta_n(z) = 2^{-\frac{n}{2}} e^{i\kappa z} (\kappa^2 v(2^{-n}z) - 2^{-(n-1)} i \kappa v'(2^{-n}z) - 2^{-2n} v''(2^{-n}z)).$$

A simple change of variables leads to

$$\begin{aligned}
 \|(\mathcal{D}_z^2 - \kappa^2)u_n\|_{L^2(\mathbb{R}^3)}^2 &\leq \int_{-\infty}^{\infty} e^{-c_n^2 x^2} c_n dx \int_{-\infty}^{\infty} e^{-c_n^2 y^2} c_n dy \int_{-\infty}^{\infty} |2^{-(n-1)} i \kappa v'(t) + 2^{-2n} v''(t)|^2 dt \\
 &= \pi \left(2^{-2n+2} \kappa^2 \|v'\|_{L^2(\mathbb{R})}^2 + 2^{-4n} \|v''\|_{L^2(\mathbb{R})}^2 \right) \rightarrow 0.
 \end{aligned}$$

Proof of (2). In order to show that $\lambda_0(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}) \geq \frac{1}{2} \mathbf{B}^2$ for $\mathbf{B} \leq \frac{1}{2}$, we define creation and annihilation operators acting on $C_0^\infty(\mathbb{R}^3)$ as follows

$$\begin{aligned}
 \mathcal{A}_x &= e^z \mathcal{D}_x - 2i \mathbf{B}_x \mathcal{D}_z & \mathcal{A}_y &= e^{-z} \mathcal{D}_y + 2i \mathbf{B}_y \mathcal{D}_z \\
 \mathcal{A}_x^\dagger &= e^z \mathcal{D}_x + 2i \mathbf{B}_x \mathcal{D}_z & \mathcal{A}_y^\dagger &= e^{-z} \mathcal{D}_y - 2i \mathbf{B}_y \mathcal{D}_z.
 \end{aligned}$$

As \mathcal{D}_x , \mathcal{D}_y and \mathcal{D}_z are symmetric on $C_0^\infty(\mathbb{R}^3)$, we have $\mathcal{A}_x^* u = \mathcal{A}_x^\dagger u$ and $\mathcal{A}_y^* u = \mathcal{A}_y^\dagger u$ for any $u \in C_0^\infty(\mathbb{R}^3)$. One easily computes the following operator identities on $C_0^\infty(\mathbb{R}^3)$

$$\begin{aligned}
 \mathcal{A}_x^\dagger \mathcal{A}_x &= e^{2z} \mathcal{D}_x^2 + 2 \mathbf{B}_x e^z \mathcal{D}_x + 4 \mathbf{B}_x^2 \mathcal{D}_z^2 \\
 \mathcal{A}_y^\dagger \mathcal{A}_y &= e^{-2z} \mathcal{D}_y^2 + 2 \mathbf{B}_y e^{-z} \mathcal{D}_y + 4 \mathbf{B}_y^2 \mathcal{D}_z^2,
 \end{aligned}$$

In particular,

$$2 \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} = \mathcal{A}_x^\dagger \mathcal{A}_x + \mathcal{A}_y^\dagger \mathcal{A}_y + (1 - 4 \mathbf{B}^2) \mathcal{D}_z^2 + \mathbf{B}^2.$$

For $u \in C_0^\infty(\mathbb{R}^3)$, we obtain

$$\left\langle \left(2 \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} - \mathbf{B}^2 \right) u, u \right\rangle_{L^2(\mathbb{R}^3)} = \|\mathcal{A}_x u\|_{L^2(\mathbb{R}^3)}^2 + \|\mathcal{A}_y u\|_{L^2(\mathbb{R}^3)}^2 + (1 - 4 \mathbf{B}^2) \|\mathcal{D}_z u\|_{L^2(\mathbb{R}^3)}^2,$$

which gives $\text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}) \subseteq [\frac{1}{2} \mathbf{B}^2, \infty)$ for $\mathbf{B}^2 \leq \frac{1}{4}$ by virtue of [Dav95, Theorem 4.3.1].

Proof of (3). Let $\mathbf{B} = \sqrt{\mathbf{B}_x^2 + \mathbf{B}_y^2} > \frac{1}{2}$, and choose $\varphi \in [0, 2\pi)$ such that

$$\mathbf{B}_x = \sin \varphi \mathbf{B} \quad \text{and} \quad \mathbf{B}_y = \cos \varphi \mathbf{B}.$$

We consider further pairs of creation and annihilation operators acting on $C_0^\infty(\mathbb{R}^3)$ as

$$\begin{aligned} \mathcal{K}_x &= e^z \mathcal{D}_x + \sin \varphi \left(\mathbf{B} - \frac{1}{2} - i \mathcal{D}_z \right) & \mathcal{K}_y &= e^{-z} \mathcal{D}_y + \cos \varphi \left(\mathbf{B} - \frac{1}{2} + i \mathcal{D}_z \right) \\ \mathcal{K}_x^\dagger &= e^z \mathcal{D}_x + \sin \varphi \left(\mathbf{B} - \frac{1}{2} + i \mathcal{D}_z \right) & \mathcal{K}_y^\dagger &= e^{-z} \mathcal{D}_y + \cos \varphi \left(\mathbf{B} - \frac{1}{2} - i \mathcal{D}_z \right). \end{aligned}$$

On $C_0^\infty(\mathbb{R}^3)$, we have $\mathcal{K}_x^* = \mathcal{K}_x^\dagger$ and $\mathcal{K}_y^* = \mathcal{K}_y^\dagger$. Moreover,

$$\begin{aligned} \mathcal{K}_x^\dagger \mathcal{K}_x &= (e^z \mathcal{D}_x + \sin \varphi \mathbf{B})^2 + \sin^2 \varphi \left(\mathcal{D}_z^2 - \mathbf{B} + \frac{1}{4} \right) \\ \mathcal{K}_y^\dagger \mathcal{K}_y &= (e^{-z} \mathcal{D}_y + \cos \varphi \mathbf{B})^2 + \cos^2 \varphi \left(\mathcal{D}_z^2 - \mathbf{B} + \frac{1}{4} \right). \end{aligned}$$

The claimed inequality now follows as in 2. from the following operator identity

$$2 \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} = \mathcal{K}_x^\dagger \mathcal{K}_x + \mathcal{K}_y^\dagger \mathcal{K}_y + \mathbf{B} - \frac{1}{4}.$$

The statement about the special case $\mathbf{B}_y = 0$ is a consequence of 4., which is proven below.

Proof of (4). We finally show that $\frac{1}{2}(|\mathbf{B}_x| + |\mathbf{B}_y|^2 - \frac{1}{4}) \in \text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}})$ whenever $|\mathbf{B}_x| > \frac{1}{2}$ by relating $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}$ to the Maass Laplacian (5.6). We consider $\mathbb{R} \times \mathbb{H}$ with coordinates $(t, x_{\mathbb{H}}, y_{\mathbb{H}})$ and metric $dt^2 + y_{\mathbb{H}}^{-2}(dx_{\mathbb{H}}^2 + dy_{\mathbb{H}}^2)$, and let $\mathcal{I}: L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R} \times \mathbb{H})$ be the isometry given by

$$(5.22) \quad \mathcal{I}u(t, x_{\mathbb{H}}, y_{\mathbb{H}}) = y_{\mathbb{H}}^{\frac{1}{2}} u(x_{\mathbb{H}}, t, \log y_{\mathbb{H}}).$$

Similar canonical transformations appear in [Dur83, Com87, IS03]. Note that \mathcal{I} maps $C_0^\infty(\mathbb{R}^3)$ onto $C_0^\infty(\mathbb{R} \times \mathbb{H})$, and the inverse is given by

$$\mathcal{I}^{-1}w(x, y, z) = e^{-\frac{1}{2}z} w(y, x, e^z).$$

Imitating the direct computation in [IS03], we obtain

$$(\mathcal{I} \mathcal{D}_z \mathcal{I}^{-1}w)(t, x_{\mathbb{H}}, y_{\mathbb{H}}) = y_{\mathbb{H}}^{\frac{1}{2}} (\mathcal{D}_z \mathcal{I}^{-1}w)(x_{\mathbb{H}}, t, \log y_{\mathbb{H}}) = y_{\mathbb{H}}^{\frac{1}{2}} \left(y_{\mathbb{H}}^{\frac{1}{2}} \mathcal{D}_{y_{\mathbb{H}}} + \frac{i}{2} y_{\mathbb{H}}^{-\frac{1}{2}} \right) w(t, x_{\mathbb{H}}, y_{\mathbb{H}}).$$

Similarly, we obtain the following operator identities on $C_0^\infty(\mathbb{R} \times \mathbb{H})$

$$\mathcal{I} \mathcal{D}_x \mathcal{I}^{-1} = \mathcal{D}_{x_{\mathbb{H}}} \quad \mathcal{I} \mathcal{D}_y \mathcal{I}^{-1} = \mathcal{D}_t \quad \mathcal{I} u(x, y, z) \mathcal{I}^{-1} = u(x_{\mathbb{H}}, t, \log y_{\mathbb{H}}),$$

where the last identity refers to multiplication by $u \in C^\infty(\mathbb{R}^3)$. Using that

$$\mathcal{I} \mathcal{D}_z^2 \mathcal{I}^{-1} = \left(y_{\mathbb{H}} \mathcal{D}_{y_{\mathbb{H}}} + \frac{i}{2} \right)^2 = y_{\mathbb{H}}^2 \mathcal{D}_{y_{\mathbb{H}}}^2 - \frac{1}{4},$$

we obtain from (5.16)

$$\mathcal{I} \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} \mathcal{I}^{-1} = \frac{1}{2} \left((y_{\mathbb{H}} \mathcal{D}_{x_{\mathbb{H}}} + \mathbf{B}_x)^2 + y_{\mathbb{H}}^2 \mathcal{D}_{y_{\mathbb{H}}}^2 \right) + \frac{1}{2} (y_{\mathbb{H}}^{-1} \mathcal{D}_t + \mathbf{B}_y)^2 - \frac{1}{8}.$$

The first summand equals $\mathcal{H}_{\mathbf{B}_x}^{\mathbb{H}}$ as in (5.6) with spectrum given in (5.7). In particular, $\lambda_0(\mathcal{H}_{\mathbf{B}_x}^{\mathbb{H}}) = \frac{1}{2} |\mathbf{B}_x|$ if $|\mathbf{B}_x| > \frac{1}{2}$. Hence, there exists a Weyl sequence $(w_n)_{n \in \mathbb{N}}$ with $w_n \in C_0^\infty(\mathbb{H})$ and $\|w_n\|_{L^2(\mathbb{H})} = 1$ such that

$$\left\| \left((y_{\mathbb{H}} \mathcal{D}_{x_{\mathbb{H}}} + \mathbf{B}_x)^2 + y_{\mathbb{H}}^2 \mathcal{D}_{y_{\mathbb{H}}}^2 - |\mathbf{B}_x| \right) w_n \right\|_{L^2(\mathbb{H})} \rightarrow 0.$$

We choose a real sequence $c_n \searrow 0$ such that $w_n(x_{\mathbb{H}}, y_{\mathbb{H}}) = 0$ if $y_{\mathbb{H}} \leq c_n^{\frac{1}{2}}$, and let $(\chi_n)_{n \in \mathbb{N}}$ be the sequence of cut-off functions given in (5.17) with $s_n = c_n^{-2}$. The claim follows once we verified that the Weyl sequence $(u_n)_{n \in \mathbb{N}}$ with elements $u_n \in C_0^\infty(\mathbb{R} \times \mathbb{H})$ given by

$$u_n(t, x_{\mathbb{H}}, y_{\mathbb{H}}) = c_n^{\frac{1}{2}} e^{-\frac{1}{2} c_n^2 t^2} \chi_n(t) w_n(x_{\mathbb{H}}, y_{\mathbb{H}})$$

satisfies $\|u_n\|_{L^2(\mathbb{R}^3)} \rightarrow \pi^{\frac{1}{4}}$ and

$$\left\| \left(\mathcal{I} \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}} \mathcal{I}^{-1} - \frac{1}{2} \left(|\mathbf{B}_x| + |\mathbf{B}_y|^2 - \frac{1}{4} \right) \right) u_n \right\|_{L^2(\mathbb{R}^3)} \rightarrow 0.$$

The first statement follows by dominated convergence as in (5.18). In order to prove the second statement, it suffices to show that

$$\|y_{\mathbb{H}}^{-k} \mathcal{D}_t u_n\|_{L^2(\mathbb{R} \times \mathbb{H})} \rightarrow 0 \quad \text{for } k = 1, 2.$$

The corresponding computations can be carried out along the lines of (5.19), (5.20) and (5.21), where one additionally uses that $y_{\mathbb{H}}^{-1} \leq c_n^{-\frac{1}{2}}$ on the support of w_n . \square

If one compares with [IS03], it appears natural to suspect that $\frac{1}{2}(\mathbf{B} - \frac{1}{4})$ is an eigenvalue of $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\text{Sol}}$ for $\mathbf{B} = \sqrt{\mathbf{B}_x^2 + \mathbf{B}_y^2} > \frac{1}{2}$. In the following, we briefly outline the spectral analysis of the magnetic Schrödinger operator (5.16) on compact quotients of Sol and their maximal abelian covers. Let $A = (A_{ij}) \in SL(2, \mathbb{Z})$ and $P = (P_{ij}) \in GL(2, \mathbb{R})$ satisfy (5.14), and denote the corresponding cocompact lattice of Sol by Λ_A . We have $A_{12} \neq 0$ as A has positive, distinct eigenvalues $\lambda^{\pm 1} = \frac{1}{2} \left(\text{Tr} A \pm \sqrt{(\text{Tr} A)^2 - 4} \right) \notin \mathbb{Q}$. Since the columns of P^{-1} are scalar multiples of the eigenvectors $(A_{12}, \lambda - A_{11})^T$ and $(A_{12}, \lambda^{-1} - A_{11})^T$ of A , and since $\lambda^{\pm 1} \notin \mathbb{Q}$, the only solution to

$$(5.23) \quad q_1 P_{11} + q_2 P_{12} = 0 \quad \text{or} \quad q_1 P_{21} + q_2 P_{22} = 0$$

with $(q_1, q_2) \in \mathbb{Q}^2$ is given by $q_1 = q_2 = 0$, which we will use later on. In order to verify that $[\Lambda_A, \Lambda_A] \backslash \text{Sol}$ is homeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, we note that $[\Lambda_A, \Lambda_A]$ is easily seen to be generated by

$$\left\{ \left(P (I - A^l) \begin{pmatrix} m \\ n \end{pmatrix}, 0 \right) \in \mathbb{R}^3 \mid (l, m, n) \in \mathbb{Z}^3 \right\}.$$

In other words, $[\Lambda_A, \Lambda_A]$ is a \mathbb{Z}^2 -subgroup of Sol, and we can find $M = (M_{ij}) \in GL(2, \mathbb{R})$ with integer entries $M_{ij} \in \mathbb{Z}$ such that

$$a = M_{11} \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} + M_{21} \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} \quad \text{and} \quad b = M_{12} \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix} + M_{22} \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}$$

form a \mathbb{Z}^2 -basis of the corresponding lattice in $\mathbb{R}^2 \simeq \mathbb{R}^2 \times \{0\} \subset \text{Sol}$. We let $a^*, b^* \in (\mathbb{R}^2)^*$ denote the corresponding dual basis vectors, that is,

$$\begin{pmatrix} a_1^* & a_2^* \\ b_1^* & b_2^* \end{pmatrix} = \text{Det}(PM)^{-1} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} \begin{pmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{pmatrix}.$$

Using the dual lattice $\mathbb{Z}a^* \oplus \mathbb{Z}b^*$, we perform discrete Fourier analysis on $[\Lambda_A, \Lambda_A] \backslash \text{Sol}$. The functions $(e_{m,n})_{m,n \in \mathbb{Z}} \subset C^\infty(\mathbb{Z}a \oplus \mathbb{Z}b \backslash \mathbb{R}^2)$ given by

$$e_{m,n}(x, y) = e^{2\pi i (ma^* + nb^*)(x, y)} = e^{2\pi i ((ma_1^* + nb_1^*)x + (ma_2^* + nb_2^*)y)}$$

yield a complete orthogonal set of smooth eigenfunctions of \mathcal{D}_x and \mathcal{D}_y in $L^2(\mathbb{Z}a \oplus \mathbb{Z}b \backslash \mathbb{R}^2, dx dy)$. On each $\mathbb{C}e_{m,n} \times C_0^\infty(\mathbb{R})$, the operator $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}}$ reduces to

$$(5.24) \quad \mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y, m, n}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}} = \frac{1}{2} \left((2\pi (ma_1^* + nb_1^*) e^z + \mathbf{B}_x)^2 + (2\pi (ma_2^* + nb_2^*) e^{-z} + \mathbf{B}_y)^2 + \mathcal{D}_z^2 \right).$$

For $m = n = 0$, we obtain $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y, 0, 0}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}} = \frac{1}{2} (\mathbf{B}_x^2 + \mathbf{B}_y^2 + \mathcal{D}_z^2)$ with purely continuous spectrum $\text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y, 0, 0}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}}) = [\frac{1}{2} \mathbf{B}^2, \infty)$. With regard to $(m, n) \neq (0, 0)$, note that, according to (5.23), we have $ma_1^* + nb_1^* = 0$, that is,

$$(-n M_{11} + m M_{12}) P_{21} + (-n M_{21} + m M_{22}) P_{22} = 0,$$

if and only if $(-n, m)$ is in the kernel of M , that is, precisely if $m = n = 0$, and similarly for $ma_2^* + nb_2^* = 0$. Hence, any $(m, n) \neq (0, 0)$ leads to a Schrödinger operator of the form $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y, m, n}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}} = \frac{1}{2} \mathcal{D}_z^2 + \mathbf{V}(z)$ with $\mathbf{V}(z) \rightarrow \infty$ for $|z| \rightarrow \infty$. Such operators are known to have purely discrete spectrum [RS78, Theorem XIII.16], and we obtain

$$\text{spec}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}}) = \left[\frac{1}{2} \mathbf{B}^2, \infty \right) \cup \overline{\bigcup_{(m,n) \neq (0,0)} \text{spec}_{\text{pp}}(\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y, m, n}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}})}.$$

The spectral analysis on the compact quotient $\Lambda_A \backslash \text{Sol}$ can be carried out as in [BDV06, Section 5], where the special case of the Laplacian $\Delta^{\Lambda_A \backslash \text{Sol}} = 2 \mathcal{H}_{0,0}^{\Lambda_A \backslash \text{Sol}}$ is considered. In particular, $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{\Lambda_A \backslash \text{Sol}}$ allows for a similar decomposition in the \mathbb{T}^2 -fibres as its lift $\mathcal{H}_{\mathbf{B}_x, \mathbf{B}_y}^{[\Lambda_A, \Lambda_A] \backslash \text{Sol}}$. We let $(w_{m,n,l})_{l \in \mathbb{N}} \subset C^\infty((\Lambda_A \cap \mathbb{R}^2 \times \{0\}) \backslash \text{Sol})$ denote the eigenfunctions of (5.24) for $M_{11} = M_{22} = 1$, $M_{12} = M_{21} = 0$ and $(m, n) \neq (0, 0)$. In general, they do not descend to functions on $\Lambda_A \backslash \text{Sol}$ since they are not invariant under $(x, y, z) \mapsto (\lambda x, \lambda^{-1} y, z + \log \lambda)$. This can be overcome by averaging, namely, one considers

$$u_{m,n,l} = \sum_{k \in \mathbb{Z}} w_{m,n,l}(\lambda^k x, \lambda^{-k} y, z + \log \lambda^k)$$

instead, where one has to use the exponential decay of the eigenfunctions $(w_{m,n,l})_{l \in \mathbb{N}}$. For $m = n = 0$, the Λ_A -invariant eigenfunctions are easily seen to be of the form $u_{0,0,l}(x, y, z) = e^{2\pi i l z (\log \lambda)^{-1}}$ with eigenvalues $\frac{1}{2} (\mathbf{B}_x^2 + \mathbf{B}_y^2 + 4\pi^2 l^2 (\log \lambda)^{-2})$. Finally, one can work along the proof of [BDV06, Theorem 2] to show that $(u_{m,n,l})_{m,n,l \in \mathbb{Z}}$ is a Hilbert basis of $L^2(\Lambda_A \backslash \text{Sol})$.

5.4.2. Monopole case. Bolsinov and Taimanov [BT00] made the remarkable discovery that the geodesic flow on compact quotients $\Lambda_A \backslash \text{Sol}$ is completely integrable in the sense of Liouville despite having non-zero topological entropy. This result motivated further study of the classical and quantum dynamics on Sol-quotients [BDV06]. In particular, Butler and Paterlain [BP08] considered the magnetic flow generated by a scalar multiple of the distinguished monopole $dx \wedge dy$ in $H^2(\Lambda_A \backslash \text{Sol}, \mathbb{R})$. They showed that as soon as the magnetic field is turned on, the flow gains positive Liouville entropy and therefore ceases being Liouville integrable. Although the monopole $dx \wedge dy$ becomes exact when lifted to the universal cover Sol, none of its primitives is bounded since $\pi_1(\Lambda_A \backslash \text{Sol}) \simeq \Lambda_A$ is solvable and therefore amenable [Pat06, Lemma 5.3]. Hence, the critical value of the respective magnetic Hamiltonian on Sol is infinite, see also [CFP10]. In the following, we show how the spectral analysis of the corresponding quantum system reduces to the study of Schrödinger operators on 2-dimensional hyperbolic space. Following [BP08], we let $\mathbf{B} \in \mathbb{R}$ and consider the left-invariant magnetic field

$$\beta = \mathbf{B} dx \wedge dy \in \Omega^2(\text{Sol}, \mathbb{R}) \quad \text{with potential} \quad \alpha = \mathbf{B} x dy \in \Omega^1(\text{Sol}, \mathbb{R}).$$

The associated magnetic Schrödinger operator $\mathcal{H}_{\mathbf{B}}^{\text{Sol}}$ acts on its initial domain $C_0^\infty(\mathbb{R}^3)$ as

$$(5.25) \quad \mathcal{H}_{\mathbf{B}}^{\text{Sol}} = \frac{1}{2} (e^{2z} \mathcal{D}_x^2 + e^{-2z} (\mathcal{D}_y + \mathbf{B} x)^2 + \mathcal{D}_z^2).$$

Proposition 29. *The spectrum of the closure of (5.25) depends on \mathbf{B} as follows:*

- (1) *We have $\text{spec}(\mathcal{H}_{\mathbf{B}}^{\text{Sol}}) \subseteq [\frac{1}{2}\mathbf{B}, \infty)$.*
- (2) *If \mathbb{H} denotes the hyperbolic upper half-plane with coordinates $(x_{\mathbb{H}}, y_{\mathbb{H}}) \in \mathbb{R} \times \mathbb{R}^+$ and metric $y_{\mathbb{H}}^{-2}(dx_{\mathbb{H}}^2 + dy_{\mathbb{H}}^2)$, then*

$$\text{spec}(\mathcal{H}_{\mathbf{B}}^{\text{Sol}}) = \text{spec}(\mathcal{H}_{0, \mathbf{V}_{\mathbf{B}}}^{\mathbb{H}} - \frac{1}{8}) \quad \text{with} \quad \mathbf{V}_{\mathbf{B}}(x_{\mathbb{H}}, y_{\mathbb{H}}) = \frac{1}{2} \mathbf{B}^2 \frac{x_{\mathbb{H}}^2}{y_{\mathbb{H}}^2}.$$

Proof of 1. We define creation and annihilation operators acting on $C_0^\infty(\mathbb{R}^3)$ as

$$\begin{aligned} \mathcal{A} &= i e^z \mathcal{D}_x + e^{-z} (\mathcal{D}_y + \mathbf{B} x) \\ \mathcal{A}^\dagger &= -i e^z \mathcal{D}_x + e^{-z} (\mathcal{D}_y + \mathbf{B} x). \end{aligned}$$

Note that $\mathcal{A}^* u = \mathcal{A}^\dagger u$ for $u \in C_0^\infty(\mathbb{R}^3)$. Since

$$\mathcal{A}^\dagger \mathcal{A} = e^{2z} \mathcal{D}_x^2 + e^{-2z} (\mathcal{D}_y + \mathbf{B} x)^2 - \mathbf{B},$$

the claim follows from $2\mathcal{H}_{\mathbf{B}}^{\text{Sol}} = \mathcal{A}^\dagger \mathcal{A} + \mathcal{D}_z^2 + \mathbf{B}$ by an application of [Dav95, Theorem 4.3.1]. We note that $\mathcal{H}_{\mathbf{B}}^{\text{Sol}}$ has the so-called translation shape invariance [IS03]

$$\mathcal{A} \mathcal{A}^\dagger + \mathcal{D}_z^2 + \mathbf{B} = \mathcal{H}_{\mathbf{B}}^{\text{Sol}} + 2\mathbf{B}.$$

Proof of 2. We reuse the isometry $\mathcal{I}: L^2(\mathbb{R}^3, dx dy dz) \rightarrow L^2(\mathbb{R} \times \mathbb{H}, y_{\mathbb{H}}^{-2} dt dx_{\mathbb{H}} dy_{\mathbb{H}})$ given in (5.22) to obtain the following operator identity on $C_0^\infty(\mathbb{R} \times \mathbb{H})$

$$\mathcal{H}_{\mathbf{B}}^{\mathbb{R} \times \mathbb{H}} = \mathcal{I} \mathcal{H}_{\mathbf{B}}^{\text{Sol}} \mathcal{I}^{-1} = \frac{1}{2} \left(y_{\mathbb{H}}^2 (\mathcal{D}_{x_{\mathbb{H}}}^2 + \mathcal{D}_{y_{\mathbb{H}}}^2) + y_{\mathbb{H}}^{-2} (\mathcal{D}_t + \mathbf{B} x_{\mathbb{H}})^2 - \frac{1}{4} \right).$$

The linear span D of $\mathcal{S}(\mathbb{R}) \times C_0^\infty(\mathbb{H})$ is easily seen to be a core for $\mathcal{H}_{\mathbf{B}}^{\mathbb{R} \times \mathbb{H}}$ by using the canonical isometry $L^2(\mathbb{R}) \otimes L^2(\mathbb{H}) \simeq L^2(\mathbb{R} \times \mathbb{H})$, cut-off functions, and dominated convergence. We

conjugate by the partial Fourier transformation $\mathcal{F}_t: D \rightarrow D$ given by

$$\mathcal{F}_t u(\xi_t, x_{\mathbb{H}}, y_{\mathbb{H}}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t, x_{\mathbb{H}}, y_{\mathbb{H}}) e^{-i\xi_t t} dt$$

to obtain a direct integral decomposition of $\mathcal{H}_{\mathbf{B}}^{\mathbb{R} \times \mathbb{H}}$ over $L^2(\mathbb{R}, d\xi_t, L^2(\mathbb{H}, y_{\mathbb{H}}^{-2} dx_{\mathbb{H}} dy_{\mathbb{H}}))$ with operators

$$\mathcal{H}_{\mathbf{B}, \xi_t}^{\mathbb{R} \times \mathbb{H}} = \frac{1}{2} \left(y_{\mathbb{H}}^2 (\mathcal{D}_{x_{\mathbb{H}}}^2 + \mathcal{D}_{y_{\mathbb{H}}}^2) + y_{\mathbb{H}}^{-2} (\xi_t + \mathbf{B} x_{\mathbb{H}})^2 - \frac{1}{4} \right)$$

acting on $C_0^\infty(\mathbb{H})$ in the $L^2(\mathbb{H})$ -fibres. For $\mathbf{B} \neq 0$, the operator $\mathcal{H}_{\mathbf{B}, \xi_t}^{\mathbb{R} \times \mathbb{H}}$ is conjugate to $\mathcal{H}_{\mathbf{B}, 0}^{\mathbb{R} \times \mathbb{H}} = \mathcal{H}_{0, \mathbf{V}_{\mathbf{B}}}^{\mathbb{H}} - \frac{1}{8}$ with respect to the isometry $\mathcal{J}: L^2(\mathbb{H}) \rightarrow L^2(\mathbb{H})$ given by $\mathcal{J}u(x_{\mathbb{H}}, y_{\mathbb{H}}) = u(x_{\mathbb{H}} - \mathbf{B}^{-1} \xi_t, y_{\mathbb{H}})$. Hence, $\text{spec}(\mathcal{H}_{0, \mathbf{V}_{\mathbf{B}}}^{\mathbb{H}} - \frac{1}{8}) = \text{spec}(\mathcal{H}_{\mathbf{B}, \xi_t}^{\mathbb{R} \times \mathbb{H}})$ for any $\xi_t \in \mathbb{R}$, which implies the statement by virtue of [RS78, Theorem XIII.85]. For $\mathbf{B} = 0$, we have Laplace operators whose spectra are explicitly given in Theorem 25 and Theorem 28. \square

5.5. The planar restricted 3-body problem. Let $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $\mathbf{B} > 0$. We study the Hamiltonian $H_{\mathbf{B}}: T^*M \rightarrow \mathbb{R}$ given by

$$H_{\mathbf{B}}(x, p) = \frac{1}{2}|p|^2 - \frac{1}{|x|} - \mathbf{B}(x_2 p_1 - x_1 p_2) = \frac{1}{2}((p_1 - \mathbf{B} x_2)^2 + (p_2 + \mathbf{B} x_1)^2) - \frac{1}{|x|} - \frac{\mathbf{B}^2}{2}|x|^2.$$

For $\mathbf{B} = 1$, the system describes the Kepler problem in rotating coordinates, that is, the planar restricted 3-body problem in a rotating frame with one of the primitives having zero mass [AFKP12, Section 3]. Since M has the isometry group $O(2)$, whose discrete subgroups produce non-compact quotients, none of the developed theorems applies. With respect to polar coordinates $(r, \varphi) \in \mathbb{R}^+ \times \mathbb{S}^1$ given by $(x_1, x_2) = r(\cos \varphi, \sin \varphi)$, the Hamiltonian reads

$$H_{\mathbf{B}}^{\text{Polar}}(r, \varphi, p_r, p_\varphi) = \frac{1}{2} \left(p_r^2 + \frac{p_\varphi^2}{r^2} \right) - \frac{1}{r} + \mathbf{B} p_\varphi.$$

Hence, $H_{\mathbf{B}}^{\text{Polar}}$ and p_φ are integrals of motion. If $p_\varphi = 0$, we have $H_{\mathbf{B}}^{\text{Polar}} = \frac{1}{2}p_r^2 - \frac{1}{r}$, which leads to $\dot{r}^2 = 2(H_{\mathbf{B}}^{\text{Polar}} + \frac{1}{r})$. Energy surfaces with $H_{\mathbf{B}}^{\text{Polar}} < 0$, respectively $H_{\mathbf{B}}^{\text{Polar}} > 0$, are easily seen to be compact, respectively non-compact, and $r(t) = (\frac{3}{\sqrt{2}}t + r(0)^{\frac{3}{2}})^{\frac{2}{3}}$ is an unbounded solution with $H_{\mathbf{B}}^{\text{Polar}} = 0$. On the other hand, $p_\varphi \neq 0$ leads to

$$H_{\mathbf{B}}^{\text{Polar}}(r, \varphi, p_r, p_\varphi) = \frac{1}{2}p_r^2 + \frac{1}{2} \left(\frac{1}{p_\varphi} - \frac{p_\varphi}{r} \right)^2 - \frac{1}{2} \frac{1}{p_\varphi^2} + \mathbf{B} p_\varphi,$$

which has no solutions with energy less than $E_0 = \mathbf{B} p_\varphi - \frac{1}{2}p_\varphi^{-2}$. The solutions with $H_{\mathbf{B}}^{\text{Polar}} = E_0$ correspond to $r = p_\varphi^2$ and $\dot{\varphi} = p_\varphi^{-3} + \mathbf{B}$. Since

$$2p_\varphi^2 (H_{\mathbf{B}}^{\text{Polar}} - \mathbf{B} p_\varphi) = p_\varphi^2 p_r^2 + \left(1 - \frac{p_\varphi^2}{r} \right)^2 - 1,$$

any orbit with $H_{\mathbf{B}}^{\text{Polar}} < \mathbf{B} p_\varphi$ must be bounded, whereas energy surfaces with $H_{\mathbf{B}}^{\text{Polar}} \geq \mathbf{B} p_\varphi$ are non-compact. In the following, we study the corresponding Schrödinger operator with initial domain $C_0^\infty(M)$, namely,

$$(5.26) \quad \mathcal{H}_{\mathbf{B}} = \frac{1}{2} \Delta^M - \frac{1}{|x|} + i \mathbf{B} \left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right).$$

Note that \mathcal{H}_B is symmetric. With respect to polar coordinates, it takes the form

$$\mathcal{H}_B^{\text{Polar}} = -\frac{1}{2} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) - \frac{1}{r} - i \mathbf{B} \frac{\partial}{\partial \varphi},$$

and has domain $C_0^\infty(\mathbb{R}^+ \times \mathbb{S}^1)$. Since $\mathcal{H}_B^{\text{Polar}}$ is spherically symmetric, we can separate variables as in [RS75, Section X.I, Example 4]. Following the proof of Theorem 26, we let $D \subset \text{Dom}(\mathcal{H}_B^{\text{Polar}})$ denote the set of finite linear combinations of products uw with $u \in C_0^\infty(\mathbb{R}^+)$ and $w \in C^\infty(\mathbb{S}^1)$. Using the canonical isometry $L^2(M) \simeq L^2(\mathbb{R}^+, r dr) \otimes L^2(\mathbb{S}^1, d\varphi)$ [RS75, Theorem II.10], we see that D is dense in $L^2(M)$. Moreover, we use the decomposition of $L^2(\mathbb{S}^1, d\varphi)$ into eigenspaces of $\frac{1}{i} \frac{\partial}{\partial \varphi}$ with eigenfunctions $e_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$ to obtain

$$L^2(M) \simeq \bigoplus_{m=-\infty}^{\infty} L_m \quad \text{where} \quad L_m = L^2(\mathbb{R}^+, r dr) \otimes \mathbb{C} e_m.$$

On each $D \cap L_m$, the operator $\mathcal{H}_B^{\text{Polar}}$ reduces to a one-dimensional Schrödinger operator acting on $C_0^\infty(\mathbb{R}^+)$ as

$$\mathcal{H}_{B,m}^{\text{Polar}} = -\frac{1}{2} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} \right) - \frac{1}{r} + \mathbf{B} m.$$

We use the isometry $\mathcal{I}: L^2(\mathbb{R}^+, r dr) \rightarrow L^2(\mathbb{R}^+, dr)$ defined by $\mathcal{I}u(r) = r^{\frac{1}{2}} u(r)$ to remove the first order derivative. On $\mathcal{I}(C_0^\infty(\mathbb{R}^+)) = C_0^\infty(\mathbb{R}^+)$, we have

$$(5.27) \quad \mathcal{H}_{B,m} = \mathcal{I} \mathcal{H}_{B,m}^{\text{Polar}} \mathcal{I}^{-1} = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{m^2 - \frac{1}{4}}{2r^2} - \frac{1}{r} + \mathbf{B} m.$$

Since $\mathcal{H}_{B,m}$ commutes with complex conjugation, a theorem by von Neumann [RS75, Theorem X.3] implies that $\mathcal{H}_{B,m}$ has self-adjoint extensions. If all $\mathcal{H}_{B,m}$ were essentially self-adjoint, then the same would hold for \mathcal{H}_B , see also [RS75, Theorem VIII.33] and [RS78, X Problem 1.(a)]. However, $\mathcal{H}_{B,0}$ has non-zero deficiency indices. We summarize the well-known properties of $\mathcal{H}_{B,m}$ in the following theorem.

Theorem 30. *Let $\mathcal{H}_{B,m}$ be the differential operator (5.27) with domain $C_0^\infty(\mathbb{R}^+) \subset L^2(\mathbb{R}^+, dr)$.*

- (1) *Each $\mathcal{H}_{B,m}$ with $m \neq 0$ is essentially self-adjoint, and the closure has eigenvalues*

$$\lambda_n(\overline{\mathcal{H}_{B,m}}) = -\frac{1}{2(n+m+\frac{1}{2})^2} + \mathbf{B} m, \quad n = 0, 1, 2, \dots$$

and essential spectrum $\text{spec}_{\text{ess}}(\overline{\mathcal{H}_{B,m}}) = [\mathbf{B} m, \infty)$.

- (2) *$\mathcal{H} = \mathcal{H}_{B,0}$ has a one-dimensional family of self-adjoint extensions \mathcal{H}_ν parametrized by $\nu \in \mathbb{R} \cup \{\infty\}$ such that*

$$\begin{aligned} \mathcal{H}_\nu &= -\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{8} \frac{1}{r^2} - \frac{1}{r}, \\ \text{Dom}(\mathcal{H}_\nu) &= \left\{ u \in L^2(\mathbb{R}^+, dr) \left| \begin{array}{l} u, u' \text{ locally absolutely continuous,} \\ -\frac{1}{2} u'' - \frac{1}{8} \frac{1}{r^2} u - \frac{1}{r} u \in L^2(\mathbb{R}^+, dr), \quad \nu u_0 = u_1 \end{array} \right. \right\}, \end{aligned}$$

with boundary values u_0 and u_1 defined as

$$u_0 = \lim_{r \searrow 0} -\frac{u(r)}{\sqrt{r} \ln r} \quad \text{and} \quad u_1 = \lim_{r \searrow 0} \frac{(u(r) + u_0 \sqrt{r} \ln r)}{\sqrt{r}}.$$

The boundary condition $u_0 = 0$ ($\nu = \infty$) gives the Friedrichs extension \mathcal{H}_∞ with eigenvalues

$$(5.28) \quad \lambda_n(\mathcal{H}_\infty) = -\frac{1}{2\left(n + \frac{1}{2}\right)^2}, \quad n = 0, 1, 2, \dots$$

All self-adjoint extensions \mathcal{H}_ν have the same essential spectrum $\text{spec}_{\text{ess}}(\mathcal{H}_\nu) = [0, \infty)$.

Proof. The statements follow from the analysis of so-called MIC-Kepler systems in \mathbb{R}^3 carried out in [Gir07, Section 3], where we have $\tilde{l} = j + \frac{\delta_1 + \delta_2}{2} = m - \frac{1}{2}$. In particular, $\mathcal{H}_{\mathbf{B},m} - \mathbf{B}m$ with $m \neq 0$ is essentially self-adjoint [Gir07, (3.15)], and the closure has eigenvalues [Gir07, (3.37)]

$$\lambda_n(\overline{\mathcal{H}_{\mathbf{B},m} - \mathbf{B}m}) = -\frac{1}{2\left(n + m + \frac{1}{2}\right)^2}, \quad n = 0, 1, 2, \dots$$

The characterization of the self-adjoint extensions of $\mathcal{H} = \mathcal{H}_{\mathbf{B},0}$ follows from [AGHKH05, Theorem D.1] for $\lambda = \frac{1}{2}$, $\gamma = -2$, $\alpha = 0$, $a = 1$ and $W = 0$. The eigenvalues (5.28) are given in [Gir07, (3.34)], and an alternative proof may be found in [AGHKH05, Theorem D.1]. In contrast to the discrete spectrum, Weyl's essential spectrum theorem [RS78, Theorem XIII.14] shows that $\text{spec}_{\text{ess}}(\mathcal{H}_\nu)$ is independent from the chosen extension [RS78, Section XIII.4, Example 5]. The inclusion $[\mathbf{B}m, \infty) \subseteq \text{spec}_{\text{ess}}(\overline{\mathcal{H}_{\mathbf{B},m}})$ follows as in [Dav95, Theorem 8.3.1] using explicit quasi-eigenfunctions and the effective potential $\mathbf{V}_{\text{eff}}(r) = \frac{m^2 - \frac{1}{4}}{2r^2} - \frac{1}{r}$, which satisfies $\mathbf{V}_{\text{eff}}(r) \rightarrow 0$ for $r \rightarrow \infty$. A detailed proof of $\inf \text{spec}_{\text{ess}}(\overline{\mathcal{H}_{\mathbf{B},m} - \mathbf{B}m}) \geq 0$ can be found in the appendix of [Few93]. \square

The eigenvalues (5.28) are commonly referred to as Bohr levels. Different values of $\nu \in \mathbb{R}$ lead to different eigenvalues often denoted as

$$\lambda_n(\mathcal{H}_\nu) = -\frac{1}{2\left(n + \frac{1}{2} - \delta\right)^2}, \quad n = 0, 1, 2, \dots$$

where δ is called Rydberg correction or quantum defect [Few93].

Corollary 31. *For any $\mathbf{B} \in \mathbb{R}$, the closure $\overline{\mathcal{H}_{\mathbf{B}}}$ has a one-dimensional family of self-adjoint extensions. The spectrum of any such extension is the entire real axis.*

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